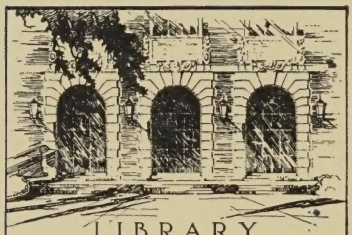

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A

TREATISE

ON

CONIC SECTIONS,

CONTAINING

AN ACCOUNT

OF SOME OF

THE MOST IMPORTANT MODERN ALGEBRAIC AND GEOMETRIC METHODS.

BY THE

REV. GEORGE SALMON,

FELLOW AND TUTOR, TRINITY COLLEGE, DUBLIN.

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CONTENTS.

The Chapters marked with an asterisk *include the principal parts of the Theory of Analytic Geometry, to which beginners should first direct their attention.*

PAGE.

*CHAPTER I.

THE POINT, ... 1
DES CARTES' METHOD OF CO-ORDINATES, ... 2
TRANSFORMATION OF CO-ORDINATES, ... 6
POLAR CO-ORDINATES, ... 8

*CHAPTER II.

THE RIGHT LINE, ... 9
ITS POLAR EQUATION, ... 27

CHAPTER III.

EXAMPLES ON THE RIGHT LINE, ... 28
EXAMPLES SOLVED BY POLAR CO-ORDINATES, ... 45

*CHAPTER IV.

ABRIDGED NOTATION APPLIED TO THE EQUATION OF THE RIGHT LINE, ... 47
TEST WHETHER THREE RIGHT LINES MEET IN A POINT, ... 50
HARMONIC AND ANHARMONIC PENCILS, ... 51
EQUATION OF LINE AT INFINITY, ... 58
TEST WHETHER RIGHT LINE PASSES THROUGH A FIXED POINT, ... 60

*CHAPTER V.

EQUATIONS REPRESENTING RIGHT LINES, ... 64
IMAGINARY RIGHT LINES, ... 65
CONDITIONS THAT AN EQUATION SHOULD REPRESENT RIGHT LINES, ... 69
NUMBER OF TERMS IN AN EQUATION OF n^{th} DEGREE, ... 70

*CHAPTER VI.

THE CIRCLE, ... 71
IMAGINARY POINTS, ... 78
POLAR EQUATION OF CIRCLE, ... 83

CHAPTER VII.

EXAMPLES ON CIRCLE, ... 84
SOLVED BY POLAR CO-ORDINATES, ... 96

685260

PAGE.

CHAPTER VIII.

ABRIDGED NOTATION APPLIED TO THE CIRCLE, ... 101
PROPERTIES OF TWO CIRCLES, ... 106
PROPERTIES OF THREE CIRCLES, ... 114

*CHAPTER IX.

DISCUSSION OF THE GENERAL EQUATION OF THE SECOND DEGREE, ... 120

*CHAPTER X.

CENTRAL EQUATIONS OF SECOND DEGREE, ... 138
THE EQUATION REFERRED TO THE AXES, ... 143
THE TANGENT, ... 149
CONJUGATE DIAMETERS, ... 151
THE ECCENTRIC ANGLE, ... 157
THE NORMAL, ... 161
THE FOCI, ... 163
THE ASYMPTOTES, ... 171

*CHAPTER XI.

THE PARABOLA, ... 176

CHAPTER XII.

EXAMPLES AND MISCELLANEOUS PROPERTIES OF CONICS, ... 188
*SIMILAR CONIC SECTIONS, ... 201
*THE CONTACT AND CURVATURE OF CONICS, ... 205

CHAPTER XIII.

METHODS OF ABRIDGED NOTATION, ... 209
ANHARMONIC PROPERTY OF CONICS PROVED, ... 218
PROPERTIES OF CONICS HAVING DOUBLE CONTACT, ... 219
TRILINEAR CO-ORDINATES, ... 224
EQUATIONS WHICH GIVE FOCAL PROPERTIES, ... 236
ENVELOPES, ... 238
GENERAL EQUATION TO TRILINEAR CO-ORDINATES, ... 242
INSCRIBED AND CIRCUMSCRIBED TRIANGLES, ... 247

CHAPTER XIV.

GEOMETRICAL METHODS, ... 251
RECIPROCAL POLARS, ... 251
HARMONIC AND ANHARMONIC PROPERTIES DEVELOPED, ... 269
INVOLUTION, ... 274
METHOD OF INFINITESIMALS, ... 288
METHOD OF PROJECTIONS, ... 297
ORTHOGONAL PROJECTIONS, ... 317

NOTES, ... 321

INDEX, ... 331

ANALYTIC GEOMETRY.

CHAPTER I.

THE POINT.

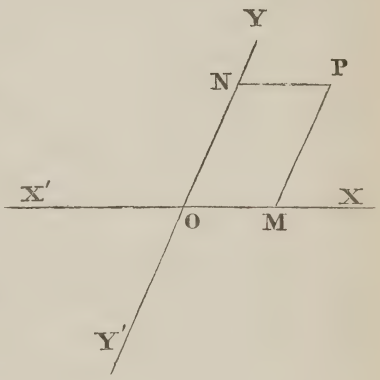
ART. 1. GEOMETRICAL theorems may be divided into two classes: theorems concerning the *magnitude* of lines, and concerning their *position;* for example, that "the square of the hypothenuse is equal to the sum of the squares of the sides," is a theorem concerning magnitude; that "the three perpendiculars of a triangle meet in a point," is a theorem concerning position.

2. Theorems of the former class can easily be expressed algebraically. To take the example already given, if the lengths of the sides of a right-angled triangle be a, b, c, the proposition alluded to is written $c^2 = a^2 + b^2$. The learner is probably already familiar with this application of algebra to geometry, as the propositions of the Second Book of Euclid all relate merely to the magnitude of lines, and the demonstration of them is much simplified by the use of algebraical symbols.

3. But it is by no means so easy to see how to express algebraically theorems involving the *position* of lines. Accordingly, although algebra was, soon after its introduction into Europe, applied to the solution of the first class of questions, its use was not extended to this latter class until the year 1637, when Des Cartes, by the publication of his "*Géométrie*," laid the foundation of the science on which we are about to enter.

4. The following method of determining the position of any point on a plane, is that introduced by Des Cartes, and generally used by succeeding geometers.

We are supposed to be given the position of two fixed right lines, XX′, YY′, intersecting in the point O. Now, if through any point P we draw PM, PN, parallel to YY′ and XX′, it is plain that, if we knew the position of the point P, we should know the lengths of the parallels PM, PN, or, *vice versâ*, that if we knew the lengths of PM, PN, we should know the position of the point O.



Suppose, for example, that we were given PN = a, PM = b, we need only measure OM = a and ON = b, and draw the parallels PM, PN, which will intersect in the point required.

It is usual to denote PM parallel to OY by the letter y, and PN parallel to OX by the letter x, and the point P would be said to be determined by the two equations $x = a$, $y = b$.

5. The parallels PM, PN, are called the *co-ordinates* of the point P; that parallel to YY′ is often called the *ordinate* of the point P; and that parallel to XX′ the *abscissa*.

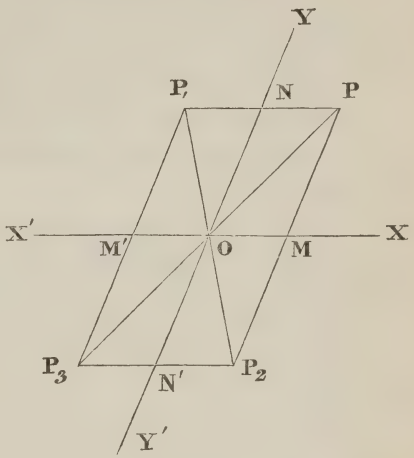
The fixed lines XX′ and YY′ are termed the *axes of co-ordinates*, and the point O, in which they intersect, is called the *origin*. The axes are said to be rectangular or oblique, according as the angle at which they intersect is a right angle or oblique.

6. In order that the equations $x = a$, $y = b$, should only be satisfied by *one* point, it is necessary to pay attention, not only to the *magnitudes*, but also to the *signs* of the co-ordinates.

If we paid no attention to the signs of the co-ordinates, we might measure OM = a and ON = b, on either side of the origin, and any of the four points, P, P_1, P_2, P_3, would satisfy the equations $x = \pm a$, $y = \pm b$.

7. It is possible, however, to distinguish algebraically between

the lines OM, OM′ (which are equal in magnitude, but opposite in direction), by giving them different signs. We lay down a rule, that if lines measured in one direction be considered as positive, lines measured in the opposite direction must be considered as negative. It is, of course, arbitrary in which direction we measure positive lines, but it is customary to consider OM (measured to the *right* hand) and ON (measured *upwards*) as positive, and OM′, ON′ (measured in the opposite directions) as negative lines.



Introducing these conventions, the four points, P, P_1, P_2, P_3, are easily distinguished. The co-ordinates of

$$\text{P are } \left.\begin{aligned} x &= +a \\ y &= +b \end{aligned}\right\},$$

$$\text{P}_1 \text{ are } \left.\begin{aligned} x &= -a \\ y &= +b \end{aligned}\right\},$$

$$\text{P}_2 \text{ are } \left.\begin{aligned} x &= +a \\ y &= -b \end{aligned}\right\},$$

$$\text{P}_3 \text{ are } \left.\begin{aligned} x &= -a \\ y &= -b \end{aligned}\right\}.$$

These distinctions of sign can present no difficulty to the learner, who is supposed to be already familiar with the principles of trigonometry.

8. A few particular cases of the application of this method of determining the position of a point deserve attention. Suppose we were asked what point is denoted by the equations $x = a$, $y = 0$?

Let us turn to the figure in Article 4, representing the general case of a point whose co-ordinates are $x = a$, $y = b$, and it is plain, that the smaller $PM = y = b$ becomes, the more nearly will P ap-

proach to the line OX, and, consequently, that the equations $x = a$, $y = 0$, must denote the point M situated *on* the axis OX, at a distance from the origin $= a$. Similarly, the equations $x = 0$, $y = b$, denote the point N, situated on the axis OY, at a distance from the origin $= b$.

The equations $x = 0$, $y = 0$, represent the origin itself, as we see by supposing $b = 0$ in the preceding example.

It appears from what has been said in the last Article, that the points

$$x = +a,\ y = +b, \text{ and } x = -a,\ y = -b,$$

lie on a right line passing through the origin; that they are equidistant from the origin, and on opposite sides of it.

N. B.—The points whose co-ordinates are $x = a$, $y = b$, or $x = x'$, $y = y'$, are generally briefly designated as the point ab, the point $x'y'$.

9. *Given the co-ordinates of two points, $x'y'$, $x''y''$, to express the distance between them, the axes of co-ordinates being supposed rectangular.*

By Euclid, I. 47,

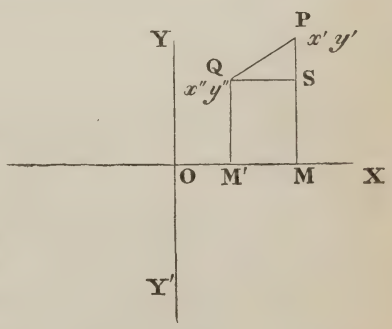
$$PQ^2 = PS^2 + SQ^2, \text{ but } PS = PM - QM' = y' - y'',$$
$$\text{and } QS = OM - OM' = x' - x'';$$

hence

$$\delta^2 = PQ^2 = (x' - x'')^2 + (y' - y'')^2.$$

To express the distance of any point from the origin, we must make $x'' = 0$, $y'' = 0$, in the above, and we find

$$\delta^2 = x'^2 + y'^2.$$



10. In the following pages we shall but seldom have occasion to make use of oblique co-ordinates, since formulæ are, in general, much simplified by the use of rectangular axes; as, however, oblique co-ordinates may sometimes be employed with advantage, we shall give the principal formulæ in their most general form.

Suppose, in the last figure, the angle YOX oblique and = ω, then

$$PSQ = 180° - \omega,$$

and (*Luby's Trig.* § 34),

$$PQ^2 = PS^2 + QS^2 - 2PS \cdot QS \cdot \cos PSQ,$$

or

$$PQ^2 = (y' - y'')^2 + (x' - x'')^2 + 2(y' - y'')\ (x' - x'') \cos\omega.$$

Similarly, the square of the distance of a point, $x'y'$, from the origin $= x'^2 + y'^2 + 2x'y' \cos\omega$.

In applying these formulæ, attention must be paid to the signs of the co-ordinates. If the point Q, for example, were in the angle XOY′, the sign of y'' would be changed, and the line PS would be the *sum* and not the *difference* of y' and y''.

11. *Given the co-ordinates of two points, $x'y'$, $x''y''$, to find the co-ordinates of the point cutting the line joining them, in a given ratio $m : n$.*

Let x, y be the co-ordinates of the point R which we seek to determine, then

$m : n :: PR : RQ :: MS : SN$,

or

$m : n :: x' - x : x - x''$,

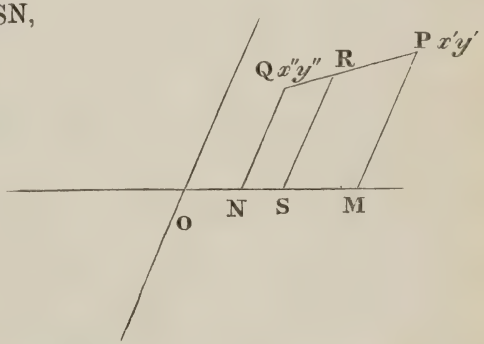
or $mx - mx'' = nx' - nx$,

hence

$$x = \frac{mx'' + nx'}{m + n}.$$

In like manner

$$y = \frac{my'' + ny'}{m + n}.$$



This includes the simpler case, to find the co-ordinates of the point bisecting the line joining two given points,

$$x = \frac{x' + x''}{2}, \quad y = \frac{y' + y''}{2}.$$

If the line were to be cut *externally* in the given ratio, we should have

$$m : n :: x - x' : x - x'',$$

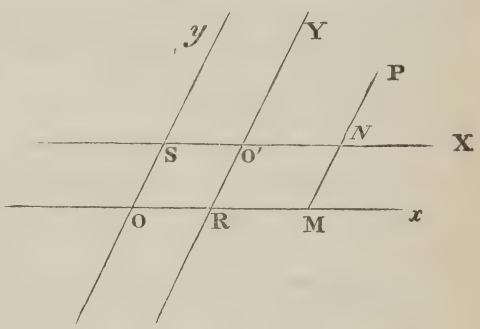
and therefore

$$x = \frac{mx'' - nx'}{m - n}, \quad y = \frac{my'' - ny'}{m - n}.$$

We can sufficiently distinguish the cases of internal and external section, if we agree that to cut a line in the ratio $+\frac{m}{n}$ shall denote to cut it *internally* in a certain ratio; and that to cut it in the ratio $-\frac{m}{n}$ shall denote to cut it *externally* in the same ratio: for the formulæ for external section are obtained from those for internal section by changing the sign of $\frac{m}{n}$.

12. When we know the co-ordinates of a point referred to one pair of axes, it is frequently necessary to find its co-ordinates referred to another pair of axes. This operation is called *the transformation of co-ordinates.*

We shall consider three cases separately: first, we shall suppose the origin changed, but the new axes parallel to the old; secondly, we shall suppose the direction of the axes changed, but the origin to remain unaltered; and thirdly, we shall examine the case when both origin and direction of the axes are altered.



First. Let the new axes be parallel to the old.

Let Ox, Oy, be the old axes, O′X, O′Y, the new axes. Let the co-ordinates of the new origin referred to the old be x', y', or O′S = x', O′R = y'. Let the old co-ordinates be x, y, the new X, Y, then we have

$$\text{OM} = \text{OR} + \text{RM}, \text{ and } \text{PM} = \text{PN} + \text{NM},$$

that is,

$$x = x' + \text{X}, \text{ and } y = y' + \text{Y}.$$

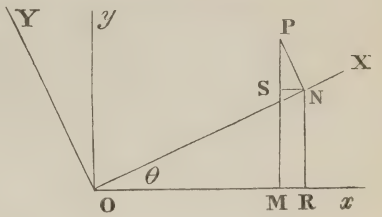
These formulæ are, evidently, equally true, whether the axes be oblique or rectangular.

13. Next, let the direction of the axes be changed, while the origin is unaltered.

(1.) We shall commence with the case where both systems are rectangular, and we shall denote by θ the angle $x\mathrm{OX} = y\mathrm{OY}$.

Then

$\mathrm{PM} = \mathrm{PS} + \mathrm{NR}$; $\mathrm{OM} = \mathrm{OR} - \mathrm{SN}$.



But since the angle

$$\mathrm{SPN} = x\mathrm{OX} = \theta,$$

$\mathrm{PS} = \mathrm{PN}\cos\theta$, $\mathrm{NR} = \mathrm{ON}\sin\theta$;

$\mathrm{OR} = \mathrm{ON}\cos\theta$, $\mathrm{SN} = \mathrm{PN}\sin\theta$.

We have, therefore,

$$y = \mathrm{Y}\cos\theta + \mathrm{X}\sin\theta, \qquad x = \mathrm{X}\cos\theta - \mathrm{Y}\sin\theta.$$

(2.) In general let the angles between the axes be any whatever. In the figure then PS, PN are drawn parallel to Oy, OY, and NS to Ox. Then, as before,

$$\mathrm{PM} = \mathrm{PS} + \mathrm{NR}.$$

We have no longer $\mathrm{PS} = \mathrm{PN}\cos\mathrm{SPN}$, since SPN is not supposed a right angle; but

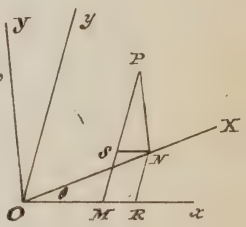
$$\mathrm{PS} : \mathrm{PN} :: \sin\mathrm{PNS}(=\sin\mathrm{YO}x) : \sin\mathrm{PSN}\ (=\sin y\mathrm{O}x);$$

$$\therefore \mathrm{PS} = \frac{\mathrm{PN}\cdot\sin\mathrm{YO}x}{\sin y\mathrm{O}x};$$

and

$$\mathrm{NR} : \mathrm{ON} :: \sin x\mathrm{OX} : \sin\mathrm{NRO}(=\sin y\mathrm{O}x),$$

$$\therefore \mathrm{NR} = \frac{\mathrm{ON}\cdot\sin x\mathrm{OX}}{\sin y\mathrm{O}x}.$$



Hence

$$y\sin x\mathrm{O}y = \mathrm{Y}\sin x\mathrm{OY} + \mathrm{X}\sin x\mathrm{OX}.$$

Similarly we find

$$x\sin y\mathrm{O}x = \mathrm{X}\sin y\mathrm{OX} + \mathrm{Y}\sin y\mathrm{OY}.$$

In using these formulæ, attention must be paid to the signs of the angles concerned in them.

The sign + is to be used when the angles $x\mathrm{O}y$, $x\mathrm{OY}$, $x\mathrm{OX}$, are all measured on the same side of Ox; and $y\mathrm{O}x$, $y\mathrm{OX}$, $y\mathrm{OY}$, on the same side of Oy.

In the case represented in the figure, the angle $y\mathrm{OY}$ lies on the opposite side of Oy from the angles $y\mathrm{O}x$ and $y\mathrm{OX}$, and the formula would become

$$x\sin y\mathrm{O}x = \mathrm{X}\sin y\mathrm{OX} - \mathrm{Y}\sin y\mathrm{OY}.$$

14. Lastly, by combining the transformations of the two preceding sections, we can find the co-ordinates of a point referred to two new axes in any position whatever. We first find the co-ordinates (by Art. 12) referred to a pair of axes through the new origin parallel to the old axes, and then (by Art. 13) we can find the co-ordinates referred to the required axes.

We thus obtain, in the case where both systems are rectangular,

$$y = y' + \mathrm{Y}\cos\theta + \mathrm{X}\sin\theta, \text{ and } x = x' + \mathrm{X}\cos\theta - \mathrm{Y}\sin\theta;$$

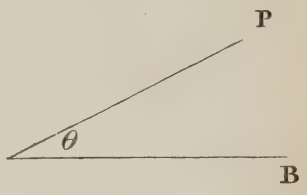
and in the more general case,

$$y \sin xOy = y' \sin xOy + \mathrm{Y}\sin xO\mathrm{Y} + \mathrm{X}\sin xO\mathrm{X},$$

and

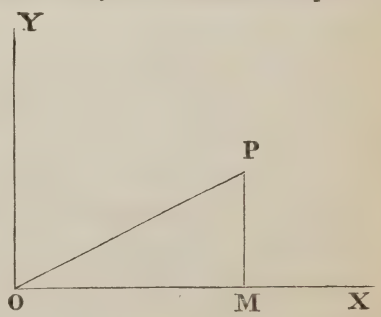
$$x \sin yOx = x' \sin yOx + \mathrm{Y}\sin yO\mathrm{Y} + \mathrm{X}\sin yO\mathrm{X}.$$

15. Beside the method of expressing the position of a point which we have hitherto made use of, there is also another which can often be employed with advantage.



If we were given a fixed point O, and a fixed line through it, OB, it is evident that we should know the position of any point, P, if we knew the length OP, and also the angle POB. The line OP is called the *radius vector;* the fixed point is called the *pole ;* and this method is called the method of *polar co-ordinates.*

It is very easy, being given the x and y co-ordinates of a point, to find its polar ones, or *vice versâ.*



First, let the fixed line coincide with the axis of x, then we have

$$\mathrm{OP} : \mathrm{PM} :: \sin\mathrm{PMO} : \sin\mathrm{POM};$$

denoting OP by ρ, POM by θ, and YOX by ω; then

$$\mathrm{PM} \text{ or } y = \frac{\rho\sin\theta}{\sin\omega}; \text{ and similarly, } \mathrm{OM} = x = \frac{\rho\sin(\omega - \theta)}{\sin\omega}.$$

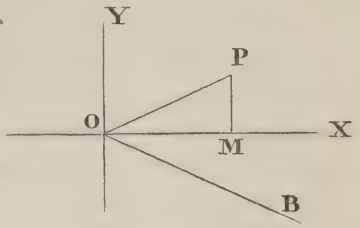
For the more ordinary case of rectangular co-ordinates, $\omega = 90°$, and we have simply

$$x = \rho \cos\theta \text{ and } y = \rho \sin\theta.$$

Secondly. Let the fixed line OB not coincide with the axis of x, but make with it an angle $= a$, then

POB $= \theta$ and POM $= \theta - a$,

and we have only to substitute $\theta - a$ for θ in the preceding formulæ.



For rectangular co-ordinates we have

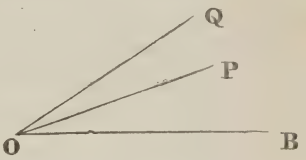
$$x = \rho \cos(\theta - a) \text{ and } y = \rho \sin(\theta - a).$$

16. *To express the distance between two points, in terms of their polar co-ordinates.*

Let P and Q be the two points,

$$OP = \rho', \quad POB = \theta';$$
$$OQ = \rho'', \quad QOB = \theta'';$$

then



$$PQ^2 = OP^2 + OQ^2 - 2OP \cdot OQ \cdot \cos POQ,$$

or

$$\delta^2 = \rho'^2 + \rho''^2 - 2\rho'\rho'' \cos(\theta'' - \theta').$$

CHAPTER II.

THE RIGHT LINE.

17. We saw, in the last chapter, that we could determine the position of a point, being given two equations regarding its co-ordinates, of the form $x = a$, $y = b$. It is evident that we could equally determine the point, had we been given any two equations of the first degree between its co-ordinates, such as

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0,$$

for we have here two equations between two variables, which we can solve by eliminating y and x alternately between them, and obtain two results of the form

$$x = a, \quad y = b.$$

18. Two equations of higher order between the co-ordinates would represent, not *one*, but a determinate number of points. For, eliminating y between the equations, we obtain an equation containing x only; let its roots be a_1, a_2, a_3, &c. Now, if we substitute any of these values (a_1) for x in the original equations, we get two equations in y, which must have a common root (since the result of elimination between the equations is rendered = 0 by the supposition $x = a_1$), Let this common root be $y = \beta_1$. Then the point whose co-ordinates are $x = a_1$, $y = \beta_1$, will at once satisfy both the given equations; and so, in like manner, will the point whose co-ordinates are $x = a_2$, $y = \beta_2$, &c.

If the given equations were of the m^{th} and n^{th} degrees respectively, the equation in x would (by the theory of elimination, see *Lacroix's Algebra*, § 196, p. 278; *Young's Algebra*, § 124, p. 229) be of the mn^{th} degree, and consequently there would be mn roots a_1, a_2, &c., and, therefore, mn points represented by the two equations.

19. A simple example will, perhaps, render this statement more clear. Let us inquire what points are represented by the two equations

$$x^2 + y^2 = 5, \qquad xy = 2.$$

In order to eliminate y between the equations, substitute for y^2, in the first equation, its value, $\frac{4}{x^2}$, derived from the second equation, and we get

$$x^4 - 5x^2 + 4 = 0.$$

The roots of this equation are $x^2 = 1$ and $x^2 = 4$, and, therefore, the four values of x are

$$x = +1, \; x = -1, \qquad x = +2, \; x = -2.$$

Substituting any of these in the second equation, we obtain the corresponding values of y,

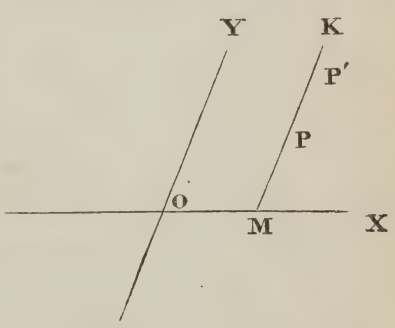
$$y = +2, \; y = -2, \qquad y = +1, \; y = -1.$$

The two given equations, therefore, represent the four points, whose co-ordinates are $(x = +1,\ y = +2)$, $(x = -1,\ y = -2)$, $(x = +2,\ y = +1)$, and $(x = -2,\ y = -1)$.

20. Having seen that any *two* equations between the co-ordinates represent geometrically one or more *points*, we proceed to inquire the geometrical signification of *a single* equation between the co-ordinates. We shall find the case to be similar to the solution of a class of geometrical problems, with which the learner is familiar. We are able to determine a triangle, being given the base and any other two conditions, but had we been given only one other condition, the vertex, though no longer determined in position, would still be limited to a certain *locus*. So we shall find, that although one equation between the two co-ordinates is not sufficient to *determine* a point, it is, however, sufficient to *limit it to a certain locus.*

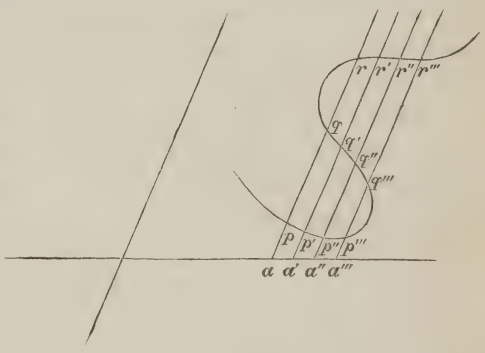
In fact, the equation asserts, that a certain relation subsists between the co-ordinates of every point represented by it. Now, although this relation will not in general subsist between the co-ordinates of any point *taken at random*, yet we shall find that there will be more points than one for which this relation *will* be true; the assemblage of these points will form a locus of points whose co-ordinates satisfy the equation, and this locus is considered the geometrical signification of the given equation.

That a single equation between the co-ordinates signifies a locus, we shall first illustrate by the simplest example. Let us recall the construction by which (Art. 4) we determined the position of a point from the two equations $x = a$, $y = b$. We took $OM = a$; we drew MK parallel to OY; and then, measuring $MP = b$, we found P, the point required. Had we been given a different value of y, $x = a$, $y = b'$, we should proceed as before, and we should find a point P′ still situated on the line MK, but at a different distance from M. Lastly, if the value of y were left wholly indeterminate, and we were merely given the



single equation $x = a$, we should know that the point P was situated *somewhere* on the line MK, but its position in that line would not be determined. Hence the line MK is the locus of all the points represented by the equation $x = a$, since, whatever point we take on the line MK, the x of that point will always $= a$.

21. In general, if we were given an equation of any degree between the co-ordinates, let us assume for x any value we please ($x = a$), and the equation will enable us to determine a finite number of values of y answering to this particular value of x, and, consequently, the equation will be satisfied for each of the points (p, q, r, &c.), whose x is the assumed value, and whose y is that found from the equation. Again, assume for x any other value ($x = a'$), and we find, in like manner, another series of points, p', q', r', whose co-ordinates satisfy the equation. So again, if we assume $x = a''$ or $x = a'''$, &c. Now, if x take successively all possible values, the assemblage of points found as above will form a *locus*, every point of which satisfies the conditions of the equation, and which is, therefore, its geometrical signification. We see then that every equation we can write down between the co-ordinates x and y must represent geometrically a locus of some kind. It is on this consideration that the whole science of Analytic Geometry is founded.



22. It is the business of Analytic Geometry to investigate the nature of the different loci represented by different equations. Then having once ascertained the locus represented by a given equation (for example, $Ax + By + C = 0$), if we find this relation subsisting between the co-ordinates of any point, we shall be sure that this point lies on the locus so determined, and, *vice versâ*, if we take any point on the locus, we shall know that this relation will exist between its co-ordinates.

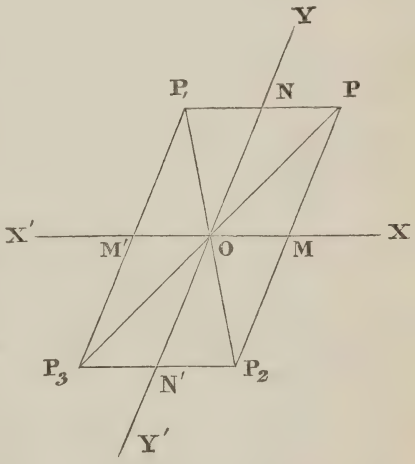
These loci are classified according to the degree of the equa-

tions representing them, being said to be of the m^{th}, n^{th}, or p^{th}, &c., degree, according as the equations representing them are of the m^{th}, n^{th}, or p^{th} degree between x and y.

We commence by examining the equation of the first degree, and we shall find that this always represents a *right line*, and, conversely, that the equation of a right line is always of the first degree.

23. We have already (Art. 20) examined the simplest case of an equation of the first degree, namely, equations of the form $x = a$, and we found that an equation of this form represents a right line parallel to the axis of y, and meeting the axis of x at a distance from the origin = to a.

In fact, if we state geometrically the conditions of the question, the locus represented by the equation $x = a$ must be such that, if from any point of it we draw PN parallel to the axis of x, to meet the axis of y, this parallel will be of a constant length = a. Now it is evident this locus is the parallel PM.



Conversely, if we examine what relation subsists between the co-ordinates of any point P, situated on a line PM, parallel to the axis OY, we see that the y of such a point is wholly indeterminate, but that the x will be always of a constant length = OM, and that, therefore, the equation $x = \text{OM}$ or $x = a$ will be satisfied for every point on the line PM.

Similarly, the equation $y = b$ represents a line PN parallel to the axis OX, and meeting the axis OY at a distance from the origin ON = b.

24. Let us now proceed to examine the case next in order of simplicity, that of a right line passing through the origin, and let us consider what relation subsists between the co-ordinates of points situated on such a line.

If we take any point P on such a line, we see that *both* the co-ordinates PM, OM, will vary in length, but that the *ratio* PM : OM will be constant, being = to the ratio

$$\sin \text{POM} : \sin \text{MPO}.$$

Hence we see, that the equation

$$y = \frac{\sin \text{POM}}{\sin \text{MPO}} x,$$

will be satisfied for every point of the line OP, and, therefore, this equation is said to be the equation of the line OP.

Conversely, if we were asked what locus was represented by the equation

$$y = mx,$$

put the equation in the form $\frac{y}{x} = m$, and the question is, "to find the locus of a point P, such that, if we draw PM, PN parallel to two fixed lines, the ratio $\frac{\text{PM}}{\text{PN}}$ may be constant." Now this locus evidently is a right line OP, passing through O, the point of intersection of the two fixed lines, and dividing the angle between them in such a manner that

$$\frac{\sin \text{POM}}{\sin \text{PON}} = m.$$

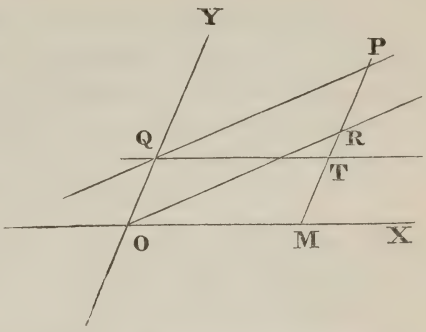
If the axes be rectangular, $\sin \text{PON} = \cos \text{POM}$, therefore, $m = \tan \text{POM}$, and the equation $y = mx$ represents a right line passing through the origin, and making an angle with the axis of x, whose tangent is m.

25. An equation of the form $y = +mx$ will denote a line OP, situated in the angles YOX, Y′OX′. On the contrary, an equation of the form $y = -mx$ will denote a line OP′, situated in the angles Y′OX, YOX′.

For it appears, from the equation $y = +mx$, that whenever x is positive y will be positive, and whenever x is negative y will be negative. Points, therefore, represented by this equation, must have their co-ordinates either both positive or both negative, and such points we saw (Art. 7) lie only in the angles YOX, Y′OX′. On the contrary, in order to satisfy the equation $y = -mx$, if x be positive y must be negative, and if x be negative y must be positive. Points, therefore, satisfying this equation, will have their

co-ordinates of *different* signs, and must, therefore (Art. 7), lie in the angles Y′OX, YOX′.

26. Let us now examine how to represent a right line PQ, situated in any manner with regard to the axes.



Draw OR through the origin parallel to PQ, and let the ordinate PM meet OR in R. Now it is plain (as in Art. 24), that the ratio RM : OM will be always constant (RM always equal, suppose, to m . OM); but the ordinate PM differs from RM by the constant length PR = OQ, which we shall call b. Hence we may write down the equation

$$\text{PM} = \text{RM} + \text{PR}, \text{ or } \text{PM} = m \,.\, \text{OM} + \text{PR},$$

that is,

$$y = mx + b.$$

The equation, therefore, $y = mx + b$, being satisfied by every point of the line PQ, is said to be the equation of that line.

It appears from the last Article, that m will be positive or negative according as OR, parallel to the right line PQ, lies in the angle YOX, or Y′OX.

Again, b will be positive or negative according as the point Q, in which the line meets OY, lies *above* or *below* the origin.

Conversely, the equation $y = mx + b$ will always denote a right line, for the equation can be put into the form

$$\frac{y - b}{x} = m.$$

Now, since if we draw the line QT parallel to OM, TM will be $= b$, and PT therefore $= y - b$, the question becomes: "To find the locus of a point, such that, if we draw PT parallel to OY to meet the fixed line QT, PT may be to QT in a constant ratio;" and this locus evidently is the right line PQ passing through Q.

The most general equation of the first degree, $Ax + By + C = 0$, can obviously be reduced to the form $y = mx + b$, since it is equivalent to

$$y = -\frac{A}{B}x - \frac{C}{B};$$

hence, the equation of the first degree *always* represents a right line.

27. From the last Articles we are able to ascertain the geometrical meaning of the constants in the equation of a right line. If the right line represented by the equation $y = mx + b$ make an angle $= \alpha$ with the axis of x, and $= \beta$ with the axis of y, then (Art. 24)

$$m = \frac{\sin \alpha}{\sin \beta};$$

and if the axes be rectangular, $m = \tan \alpha$.

We saw (Art. 26) that b is the intercept which the line cuts off on the axis of y.

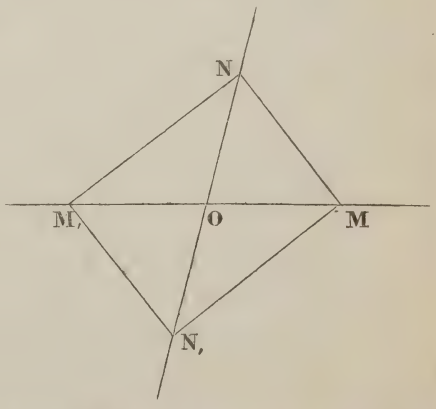
If the equation be given in the general form $Ax + By + C = 0$, we can reduce it, as in the last Article, to the form $y = mx + b$, and we find that

$$-\frac{A}{B} = \frac{\sin \alpha}{\sin \beta},$$

or if the axes be rectangular $= \tan \alpha$, and that $-\frac{C}{B}$ is the length of the intercept made by the line on the axis of y.

Beside the forms $Ax + By + C = 0$ and $y = mx + b$, there are two other remarkable forms in which the equation of a right line is frequently used; these we next proceed to lay before the reader.

28. *To find the lengths of the intercepts which the line* MN, *whose equation is* $Ax + By + C = 0$, *cuts off on the axes.*



We found in the last Article the length of one of these intercepts, by comparing the present equation with the equation $y = mx + b$. We prefer, however, in the present Article, to investigate the same question directly, by the help of an important principle already alluded to (Art. 22). The co-ordinates of *every* point of the line MN must of course satisfy the

given equation, therefore so must the co-ordinates of the point M, where this line meets the axis of x. Now for every point on the axis of x, $y = 0$ (Art. 8), therefore, for the point M, the equation gives $Ax + C = 0$, but the x of the point M is the intercept OM, whose length is required; therefore,

$$OM = -\frac{C}{A}.$$

Similarly,

$$ON = -\frac{C}{B}.$$

Hence it is easy to find the equation of a line which shall cut off intercepts on the axes, $OM = a$ and $ON = b$.

The general equation of a right line is

$$Ax + By + C = 0, \text{ or } \frac{A}{C}x + \frac{B}{C}y + 1 = 0;$$

but

$$\frac{A}{C} = -\frac{1}{OM} = -\frac{1}{a}, \text{ and } \frac{B}{C} = -\frac{1}{ON} = -\frac{1}{b};$$

therefore, the equation of the right line required is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

This is the equation of the right line in terms of the intercepts it cuts off on the axes.* It evidently holds whether the axis be oblique or rectangular.

It is plain that the position of the line will depend on the signs of the quantities a and b. For example, the given equation $\frac{x}{a} + \frac{y}{b} = 1$, which cuts off positive intercepts on both axes, represents the line MN on the preceding figure;

$\frac{x}{a} - \frac{y}{b} = 1$, cutting off a positive intercept on the axis of x, and a negative intercept on the axis of y, represents MN′.

Similarly,

$$\frac{y}{b} - \frac{x}{a} = 1 \text{ represents NM′;}$$

and

$$\frac{x}{a} + \frac{y}{b} = -1 \text{ represents M′N′.}$$

* This equation is also sometimes considered as the equation of the base of a triangle, the sides being the axes of co-ordinates.

The student will find no difficulty in examining for himself how changes in the signs of A, B, or C affect the position of the line represented by the general equation

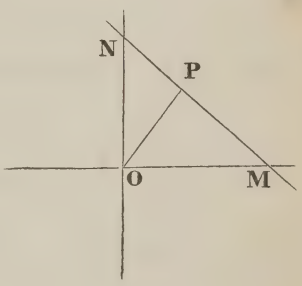
$$Ax + By + C = 0.$$

29. If we suppose A = 0 in the general equation, the intercept $-\frac{C}{A}$ made by the line on the axis of x becomes infinite. Hence the line $By + C = 0$ cuts the axis of x at an infinite distance, or, in other words, is parallel to it. This agrees with the result of Art. 23.

The distance from the origin at which this parallel meets the axis of y (Art. 27), is $-\frac{C}{B}$. If, therefore, C = 0, this distance will vanish, and the equation $y = 0$ represents the axis of x itself.

Similarly, $Ax + C = 0$ denotes a line parallel to the axis of y, and $x = 0$ the axis of y itself.

30. It is often convenient to express the equation of a right line in terms of the length of the perpendicular on it from the origin, and of the angles which this perpendicular makes with the axes.



Let the length of the perpendicular OP = p, the angle POX which it makes with the axis of $x = \alpha$, POY = β, OM = a, ON = b.

We saw (Art. 28) that the equation of the right line MN was

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Multiply this equation by p, and we have

$$\frac{p}{a}x + \frac{p}{b}y = p.$$

But

$$\frac{p}{a} = \cos\alpha; \qquad \frac{p}{b} = \cos\beta.$$

Therefore the equation of the line is

$$x\cos\alpha + y\cos\beta = p.$$

In rectangular co-ordinates, which we shall most generally use, the angle which the perpendicular makes with the axis of x will be the complement of that which it makes with the axis of y, or $\beta = 90° - \alpha$. Hence, $x \cos \alpha + y \sin \alpha = p$ is the equation referred to rectangular co-ordinates of a line, the perpendicular on which from the origin makes an angle $= \alpha$ with the axis of x, and is in length $= p$.

If we had been given the equation of a right line in the general form $Ax + By + C = 0$, it is easy to reduce it to the form $x \cos \alpha + y \sin \alpha = p$; for, divide the first by $\sqrt{(A^2 + B^2)}$, and we have

$$\frac{A}{\sqrt{(A^2 + B^2)}} x + \frac{B}{\sqrt{(A^2 + B^2)}} y + \frac{C}{\sqrt{(A^2 + B^2)}} = 0.$$

Now we may take

$$\frac{A}{\sqrt{(A^2 + B^2)}} = \cos \alpha, \text{ and } \frac{B}{\sqrt{(A^2 + B^2)}} = \sin \alpha,$$

since these two quantities fulfil the condition

$$\cos^2 \alpha + \sin^2 \alpha = 1.$$

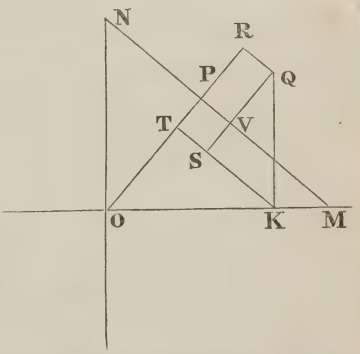
Hence we learn, that $\frac{A}{\sqrt{(A^2 + B^2)}}$ and $\frac{B}{\sqrt{(A^2 + B^2)}}$ are respectively the cosine and the sine of the angle which the perpendicular from the origin on the right line, whose equation is $Ax + By + C = 0$, makes with the axis of x, and that $\frac{-C}{\sqrt{(A^2 + B^2)}}$ is the length of this perpendicular.

31. *To find the length of the perpendicular from any point $x'y'$, on the line whose equation, referred to rectangular co-ordinates, is*

$$x \cos \alpha + y \sin \alpha - p = 0.$$

We shall show that it is found by substituting the co-ordinates x', y', for x and y in the given equation, and is equal to

$$x' \cos \alpha + y' \sin \alpha - p.$$



For, from the given point Q draw QR parallel to the given line, and QS perpendicular. Then OK $= x'$, and OT will be $= x' \cos \alpha$.

Again, since

$$\text{SKQ} = \alpha, \text{ and } \text{QK} = y',$$

$$\text{RT} = \text{QS} = y' \sin \alpha;$$

hence $$x' \cos \alpha + y' \sin \alpha = \text{OR}.$$

Subtract OP the perpendicular from the origin, and

$x' \cos \alpha + y' \sin \alpha - p = \text{PR} =$ the perpendicular QV. Q.E.D.

If the equation of the line had been given in the form $Ax + By + C = 0$, we have only to reduce it to the form

$$x \cos \alpha + y \sin \alpha - p = 0,$$

and the length of the perpendicular from any point $x'y'$,

$$= \frac{Ax' + By' + C}{\sqrt{(A^2 + B^2)}}.$$

The square root in this value is, of course, susceptible of a double sign. If we give to the perpendiculars on one side of the line the sign +, we must give to perpendiculars on the other side the sign −.

If the equation of a line referred to *oblique* co-ordinates had been $x \cos \alpha + y \cos \beta - p$, it can be proved, precisely as above, that the length of the perpendicular on it from any point is

$$x' \cos \alpha + y' \cos \beta - p.$$

The condition that any point $x'y'$, should be on the right line $Ax + By + C = 0$, is, of course, that the co-ordinates x', y', should satisfy the given equation, or

$$Ax' + By' + C = 0.$$

And the present Article shows that this condition is merely the algebraical statement of the fact, that the perpendicular from the point $x'y'$ on the given line is = 0.

32. *To find the equation of a right line passing through a given point $x'y'$.*

The general equation of a right line, we have seen, can be put under the form $y = mx + b$, where m and b are as yet unknown, and are to be determined by any conditions we are given respecting the line.

Now suppose a point on the line given, the equation $y = mx + b$, which is true for *every* point on the line, must be true for the

point $x'y'$. Hence we get the condition $y' = mx' + b$. As we are given no other condition, we are not able to determine *both* the unknown quantities m and b, but by means of this condition we can determine *one* of them, $b = y' - mx'$. Substituting this value in the general equation, we get

$$y = mx + y' - mx',$$

or

$$y - y' = m(x - x'),$$

for the equation of a right line passing through the point $x'y'$. m remains indeterminate, as it ought, since an infinite number of lines can be drawn through the point $x'y'$.

33. *To find the equation of a right line passing through* two *given points*, $x'y'$, $x''y''$.

The condition that the right line must pass through a second point will now enable us to determine the constant m which was left indeterminate in the last Article.

By the last Article the equation of a right line through $x'y'$ is

$$y - y' = m(x - x'),$$

or

$$\frac{y - y'}{x - x'} = m.$$

But since the line must also pass through the point $x''y''$, this equation must be satisfied when the co-ordinates x'', y'' are substituted for x and y; hence

$$\frac{y'' - y'}{x'' - x'} = m.$$

Substituting this value of m, the equation of the line becomes

$$\frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'}.$$

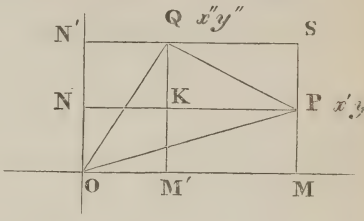
In this form the equation can be easily remembered, but, clearing it of fractions, we obtain it in a form which is sometimes more convenient,

$$(y' - y'')x - (x' - x'')y + x'y'' - y'x'' = 0.$$

It is important to observe that we can sometimes write down, without calculation, the equation of the line joining two points. If we happen to know beforehand that the co-ordinates of both points are connected by the relations $Ax' + By' + C = 0$ and

$Ax'' + By'' + C = 0$, then it is evident that the equation of the line joining them is $Ax + By + C = 0$, for it is the equation of a right line, and is satisfied by the co-ordinates of both points.

34. It is worth while to examine the geometric meaning of the terms of the equation given in the last Article. It is evident that $y'' - y' = QK$ and $x' - x'' = PK$, and we shall prove that the triangle OPQ is half $x'y'' - y'x''$ (multiplied by the sine of the angle between the axes, if they be oblique). For the triangle OPQ is



$$= OKQ + OKP + PKQ = \tfrac{1}{2}NQ + \tfrac{1}{2}MK + \tfrac{1}{2}KS = \tfrac{1}{2}OS - \tfrac{1}{2}OK,$$

but the parallelogram

$$OS = x'y'' \sin MON,$$

and

$$OK = y'x'' \sin MON.$$

35. *To find the condition that three points shall lie on one right line.*

We found (in Art. 33) the equation of the line joining two of them, and we have only to see if the co-ordinates of the third will satisfy this equation.

The condition, therefore, is

$$(y_1 - y_2)x_3 - (x_1 - x_2)y_3 + (x_1y_2 - x_2y_1) = 0,$$

which can be put into the more symmetrical form,

$$y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2) = 0.*$$

If the three points be not on one right line, the left-hand side of this equation (multiplied by $\sin\omega$, if the axes be oblique), is double the area of the triangle formed by the three points.

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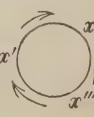
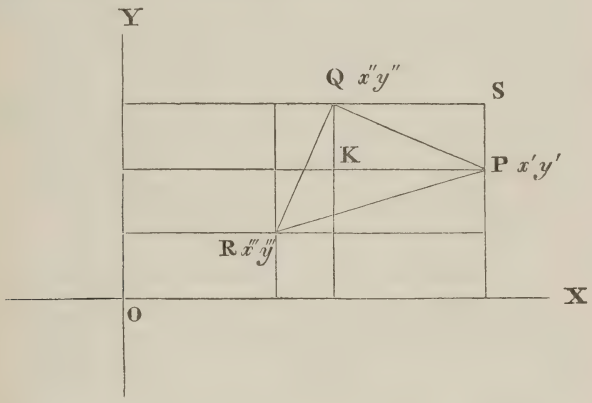
* In using this and other similar formulæ, which we shall afterwards have occasion to employ, the learner must be careful to take the co-ordinates in a fixed order (see engraving). For instance, in the second member of the formula just given y_2 takes the place of y_1, x_3 of x_2, and x_1 of x_3. Then, in the third member, we advance from y_2 to y_3, from x_3 to x_1, and from x_1 to x_2, always proceeding in the order just indicated.



For, putting it into the form

$$(x_1 - x_3)\,(y_2 - y_3) - (x_2 - x_3)\,(y_1 - y_3),$$

it is = parallelogram RS − parallelogram RK; and we can prove, precisely as in the last Article, that the difference of these parallelograms is double the triangle PQR.



Thus it appears, that the condition that three points should be on one right line, when interpreted geometrically, asserts that the area of the triangle formed by the three points becomes = 0.

36. *To express the area of a polygon in terms of the co-ordinates of its angular points.*

Take any point xy within the polygon, and connect it with all the vertices x_1y_1, x_2y_2, . . . x_ny_n; then evidently the area of the polygon is the sum of the areas of all the triangles into which the figure is thus divided. But by the last Article these areas are respectively

$$\begin{aligned}
&x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1,\\
&x(y_2 - y_3) - y(x_2 - x_3) + x_2y_3 - x_3y_2,\\
&x(y_3 - y_4) - y(x_3 - x_4) + x_3y_4 - x_4y_3,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&x(y_{n-1} - y_n) - y(x_{n-1} - x_n) + x_{n-1}y_n - x_ny_{n-1},\\
&x(y_n - y_1) - y(x_n - x_1) + x_ny_1 - x_1y_n.
\end{aligned}$$

When we add these together, the parts which multiply x and y vanish, as they evidently ought to do, since the value of the total area must be independent of the manner in which we divide it into triangles; and we have for the area

$$(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_4 - x_4 y_3) + \ldots (x_n y_1 - x_1 y_n).$$

This may be otherwise written,

$$x_1(y_2 - y_n) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + \ldots x_n(y_1 - y_{n-1}),$$

or else

$$y_1(x_n - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_4) + \ldots y_n(x_{n-1} - x_1).$$

37. *To find the co-ordinates of the point of intersection of two right lines whose equations are given.*

Each equation expresses a relation which must be satisfied by the co-ordinates of the point required; we find its co-ordinates, therefore, by solving for the two unknown quantities x and y, from the two given equations. Let the equations be given in the most general form,

$$Ax + By + C = 0, \qquad A'x + B'y + C' = 0,$$

then x will be found $= \dfrac{BC' - B'C}{AB' - BA'}$, and $y = \dfrac{AC' - CA'}{BA' - AB'}$.

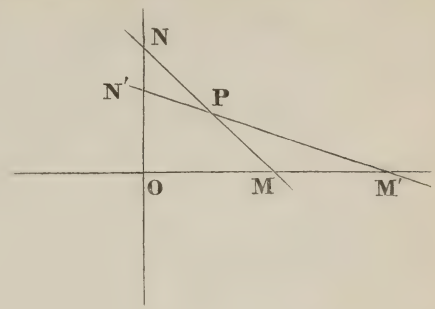
We said (Art. 17) that the position of a point was determined, being given two equations between its co-ordinates. The reader will now perceive that each equation represents a locus on which the point must lie, and that the point is the intersection of the two loci represented by the equations. Even the simplest equations to represent a point, viz., $x = a$, $y = b$, are the equations of two parallels to the axes of co-ordinates, the intersection of which is the required point.

The reader will also now understand why two equations of the first degree only represent one point, and why two equations of higher degree represent more points than one (Art. 18). In the first case each equation represents a right line, and two right lines can only intersect in one point. In the more general case, the loci represented by the equations are curves of higher dimensions, which will intersect each other in more points than one.

38. *To find the angle between two lines, whose equations with regard to rectangular co-ordinates are given.**

* All formulæ involving angles become much more complicated by the use of oblique co-ordinates. Since, therefore, in the solution of questions in which angles are concerned, we shall always be careful to choose rectangular axes, we do not think it necessary to extend the following formulæ to the case of oblique axes.

The angle between the lines is equal to the difference between the angles which each makes with the axis of x.



But if the equations be given in the form

$$y = mx + b, \quad y = m'x + b'.$$

We saw (Art. 27) that

$$m = \tan \mathrm{PMO},$$

and $$m' = \tan \mathrm{PM'O};$$

hence (*Luby's Trig.* § 15),

$$\tan \mathrm{MPM'} = \frac{m - m'}{1 + mm'}.$$

From the above expression it is easy to see that the two lines $y = mx + b$, $y = m'x + b'$ are parallel, when

$$m = m',$$

and are perpendicular to each other when

$$mm' = -1,$$

since the tangent of the angle between them vanishes in the first case, and becomes infinite in the second.

If the equations had been given in the form

$$Ax + By + C = 0, \qquad A'x + B'y + C' = 0,$$

we can at once reduce them to the form

$$y = mx + b, \qquad y = m'x + b',$$

and we have only to write in the preceding formula,

$$\text{for } m, -\frac{A}{B}; \text{ and for } m', -\frac{A'}{B'};$$

or we may solve the question directly as follows:

The angle between the lines is manifestly equal to the angle between the perpendiculars on the lines from the origin; if therefore these perpendiculars make with the axis of x the angles a, a', we have

$$\sin \mathrm{MPM'} = \sin(a - a') = \sin a \cos a' - \sin a' \cos a,$$
$$\cos \mathrm{MPM'} = \cos(a - a') = \cos a \cos a' + \sin a \sin a'.$$

But (Art. 30),

$$\cos\alpha = \frac{A}{\sqrt{(A^2+B^2)}}; \quad \sin\alpha = \frac{B}{\sqrt{(A^2+B^2)}};$$

$$\cos\alpha' = \frac{A'}{\sqrt{(A'^2+B'^2)}}; \quad \sin\alpha' = \frac{B'}{\sqrt{(A'^2+B'^2)}};$$

Hence

$$\sin \mathrm{MPM}' = \frac{BA' - AB'}{\sqrt{(A^2+B^2)}\ \sqrt{(A'^2+B'^2)}};$$

$$\cos \mathrm{MPM}' = \frac{AA' + BB'}{\sqrt{(A^2+B^2)}\ \sqrt{(A'^2+B'^2)}};$$

and therefore

$$\tan \mathrm{MPM}' = \frac{BA' - AB'}{AA' + BB'}.$$

The two lines are therefore parallel when $BA' - AB' = 0$, that is, when

$$\frac{A}{A'} = \frac{B}{B'}.$$

They are perpendicular to each other when $AA' + BB' = 0$.

39. *To find the equation of a line making a given angle, ϕ, with a given line $y = mx + b$ (the axes of co-ordinates being rectangular).*

Let the equation of the required line be

$$y = m'x + b'.$$

The formula of the last Article,

$$\tan\phi = \frac{m - m'}{1 + mm'},$$

enables us to determine

$$m' = \frac{m - \tan\phi}{1 + m\tan\phi};$$

b' remains indeterminate, as we should expect, since several parallel right lines can be drawn making the given angle with the given line.

By adding another condition we can determine b'.

For example, we can find the equation of a line *passing through a given point*, and making a given angle with the given line.

The equation of any line through $x'y'$ is (Art. 32) of the form

$$y - y' = m'(x - x'),$$

and giving to m' the value found in the preceding part of the Article, we get the equation of the required line,

$$y - y' = \frac{m - \tan\phi}{1 + m\tan\phi}(x - x'),$$

or

$$y - y' = \frac{m\cos\phi - \sin\phi}{\cos\phi + m\sin\phi}(x - x').$$

To find the equation of a right line passing through a given point, and *perpendicular* to a given line, $y = mx + b$, we have only to suppose $\cos\phi = 0$, $\sin\phi = 1$, in the above formula, and we get

$$y - y' = -\frac{1}{m}(x - x');$$

or this equation might have been obtained directly from the condition that two right lines should cut at right angles, namely, $mm' = -1$.

It is easy, from the above, to see that the equation of the perpendicular from the point $x'y'$ on the line $Ax + By + C = 0$ is

$$A(y - y') = B(x - x').$$

40. *To find the polar equation of a right line* (see Art. 15).

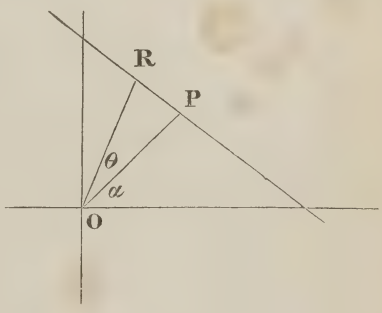
Suppose we take, as our fixed axis, the perpendicular on the given line, then let OR be any *radius vector* drawn from the pole to the given line

$$OR = \rho, \quad ROP = \theta,$$

but, plainly,

$$OR\cos\theta = OP,$$

hence, the equation is



$$\rho\cos\theta = p.$$

If the fixed axis make an angle α with the perpendicular, the equation is

$$\rho\cos(\theta - \alpha) = p.$$

This equation may also be obtained by transforming the equation with regard to rectangular co-ordinates,

$$x\cos\alpha + y\sin\alpha = p.$$

Rectangular co-ordinates are transformed to polar by writing for x, $\rho \cos\theta$, and for y, $\rho \sin\theta$ (see Art. 15); hence the equation becomes

$$\rho(\cos\theta \cos\alpha + \sin\theta \sin\alpha) = p;$$

or, as we got before,

$$\rho \cos(\theta - \alpha) = p.$$

An equation of the form

$$\rho \,.\, (A \cos\theta + B \sin\theta) = C$$

can be (as in Art. 30) reduced to the form $\rho \,.\, \cos(\theta - \alpha) = p$, by dividing by $\sqrt{(A^2 + B^2)}$; we shall then have

$$\cos\alpha = \frac{A}{\sqrt{(A^2 + B^2)}}, \quad \sin\alpha = \frac{B}{\sqrt{(A^2 + B^2)}}, \quad \tan\alpha = \frac{B}{A},$$

$$p = \frac{C}{\sqrt{(A^2 + B^2)}}.$$

CHAPTER III.

EXAMPLES ON THE RIGHT LINE.

41. Having laid down principles by which we are able to express algebraically the position of any point or right line, we proceed to give examples of the application of this method to the solution of geometrical problems. The learner should diligently exercise himself in working out such questions, until he has acquired quickness and readiness in the use of this method. The examples given in this chapter being introduced, not so much for their own sake, as to show how such questions may be solved algebraically, will, for the most part, be such as admit of simpler *geometrical* solutions. It must not be supposed, however, that because in these instances the geometrical method has the advantage, it is in all cases to be preferred. Each method has its peculiar recommendations. If the geometrical solutions of some questions are clearer and more simple, the algebraical method proceeds with more uniformity, and reaches its end with greater certainty. It should be the student's aim to make himself master of both in-

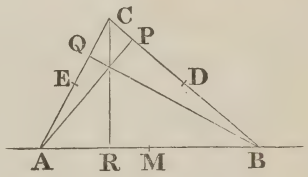
struments of investigation, so as to be able to apply either, according as the nature of the subject demands.

In the next chapter we shall give simple algebraical demonstrations of some theorems which we have here preferred to prove by the most obvious methods of investigation.

42. Ex. 1. *To prove that the three perpendiculars of a triangle meet in a point.*

In order to investigate algebraically whether three lines meet in a point, we shall form the equations of the three lines, then, by Art. 36, find the co-ordinates of the point of intersection of any two of them, and then examine whether these co-ordinates satisfy the equation of the third line. Although the *position* of the axes of co-ordinates does not affect the *validity* of this process, yet a judicious choice of axes may very much increase the *simplicity* of the equations which will occur.

In any question in which the consideration of *angles* is involved, it will be advisable to choose rectangular axes; but if the question be not of this nature, we may take for axes any two important lines on our figure, even if the angle between them be not right.



In the present case we take one of the perpendiculars CR for the axis of y, and the base for the axis of x. Let us call the length CR, p, and the segments BR, AR, s and s', then the equation of BC (whose intercept on the axis of $y = p$, and on the axis of $x = s$) is (Art. 28)

$$\frac{x}{s} + \frac{y}{p} = 1.$$

Now, the equation of a line through any point $x'y'$, perpendicular to BC, is (Art. 39)

$$\frac{1}{p}(x - x') = \frac{1}{s}(y - y').$$

Therefore, the equation of the perpendicular AP, passing through the point A, whose $y' = 0$, and whose $x' = -s'$, is

$$\frac{1}{p}(x + s') = \frac{1}{s}y.$$

In like manner the equation of AC is

$$\frac{y}{p} - \frac{x}{s'} = 1.$$

(We use the sign – since the intercept AR is measured on the negative side of the origin R.)

Therefore, the equation of BQ perpendicular to AC, and passing through B, whose $y' = 0$, and whose $x' = s$, is (Art. 39)

$$\frac{1}{p}(x - s) + \frac{1}{s'}y = 0.$$

From this equation, and that of AP,

$$\frac{1}{p}(x + s') - \frac{1}{s}y = 0,$$

we can determine the x and y of their point of intersection. Subtract them, and we get

$$\frac{s + s'}{p} = \left(\frac{1}{s} + \frac{1}{s'}\right)y, \quad \text{or } y = \frac{ss'}{p}.$$

Multiply the first by s', and the second by s, and add them, and we get

$$x = 0.$$

Now $x = 0$ is the equation of the axis of y, or of the perpendicular CR; hence we see that the perpendiculars AP and BQ intersect on CR, and at a height above the base $= \frac{ss'}{p}$.

Ex. 2. *The three bisectors of the sides of a triangle meet in a point.*

Let us retain the same axes as in the last Example.

The distance MR of the middle point of the base from the origin $= \frac{s - s'}{2}$.

Hence (Art. 28) the equation of MC is

$$\frac{2x}{s - s'} + \frac{y}{p} = 1.$$

The co-ordinates of D, the middle point of BC, are (Art. 11) $x = \frac{s}{2}$, $y = \frac{p}{2}$. And the co-ordinates of A we saw were $x = -s'$, $y = 0$.

Forming, therefore, by Art. 33, the equation of the line joining these points, we find for the equation of AD,

$$\frac{p}{2}x - \left(\frac{s}{2} + s'\right)y + \frac{ps'}{2} = 0.$$

In like manner BE, which joins the points $B(x = s,\ y = 0)$ to $E\left(x = -\frac{s'}{2},\ y = \frac{p}{2}\right)$, will have for its equation

$$\frac{p}{2}x + \left(s + \frac{s'}{2}\right)y - \frac{sp}{2} = 0.$$

In order to find the co-ordinates of the point of intersection of the last two lines, subtract their equations, and we get

$$\frac{3}{2}(s+s')y - \frac{1}{2}(s+s')p = 0, \text{ or } y = \frac{p}{3}.$$

Substitute this value of y in either equation, and we get

$$x = \frac{s - s'}{3}.$$

Now it will be seen that these values of x and y will satisfy the equation of MC,

$$\frac{2x}{s-s'} + \frac{y}{p} = 1.$$

Hence the three bisectors meet in a point. They also trisect each other in this point, as may be seen by forming (Art. 11) the co-ordinates of the point of trisection of any of them, when values will be obtained identical with those just given.

Ex. 3. The same question may furnish us with an example of the occasional use of oblique co-ordinates. We might, for example, have taken the middle point of base for origin, the base for axis of x, and the bisector MC for axis of y. Let us call the length MC, β, and let the half base $MB = c$; then AD joins the point $A(x = -c,\ y = 0)$ to $D\left(x = \frac{c}{2},\ y = \frac{\beta}{2}\right)$. Its equation, therefore, is

$$3cy - \beta x - c\beta = 0.$$

In like manner, BE joins $B(x = c,\ y = 0)$ to $E\left(x = -\frac{c}{2},\ y = \frac{\beta}{2}\right)$, and its equation is

$$3cy + \beta x - c\beta = 0.$$

To find the co-ordinates of the intersection of these lines add the equations, and we get

$$y = \frac{\beta}{3}.$$

Subtract them, and we get

$$x = 0.$$

This proves that the two lines intersect on the axis of y, that is, on the third bisector, and at a distance from the base equal to its third part.

Ex. 4. *The three perpendiculars through the middle points of the sides of a triangle meet in a point.*

We shall leave this example to be worked out by the learner, He may take the same axes as in Ex. 1. He will find given in that Example the equations of the sides of the triangle AC and BC. He must then, by Art. 39, form the equations of perpendiculars to these lines through the points D and E, whose co-ordinates are given in Ex. 2. He must then seek the co-ordinates of the point of intersection of these perpendiculars, and he will find for the x of this point,

$$x = \frac{s - s'}{2}.$$

But this proves that the perpendicular from the point of intersection on the base will pass through the middle point of base, since we saw (Ex. 2) that the distance of the middle point of base from the origin $= \frac{s - s'}{2}$.*

Ex. 5. *The three middle points of the diagonals of a complete quadrilateral lie in one right line.*

We leave this Example also to be worked out by the learner. He may take two diagonals for axes, and if the segments of these diagonals be a, a', b, b', then, as in Ex. 2, he will have for one middle point $x = \frac{a - a'}{2}$; for another, $y = \frac{b - b'}{2}$; and for the line joining them,

$$\frac{2x}{a - a'} + \frac{2y}{b - b'} = 1.$$

* For a proof that the three bisectors of the angles of a triangle meet in a point, see next chapter (Art. 57).

He can then form (by Art. 28) the equations of the four sides; thence derive (by Art. 37) the co-ordinates of the points of intersection of opposite sides; thence (by Art. 11) form the co-ordinates of the middle point of the line joining these intersections; and finally verify that these co-ordinates satisfy the equation just given.

43. *To find in what ratio the line joining two given points* $x'y'$, $x''y''$, *is cut by a given line* $(Ax + By + C)$.

Let the required ratio be $m : n$; then (Art. 11) the co-ordinates of the point of section are

$$x = \frac{mx'' + nx'}{m + n}, \quad y = \frac{my'' + ny'}{m + n}.$$

But, by hypothesis these co-ordinates satisfy the equation of the given line; therefore

$$A\,\frac{mx'' + nx'}{m + n} + B\,\frac{my'' + ny'}{m + n} + C = 0;$$

hence

$$\frac{m}{n} = -\frac{Ax' + By' + C}{Ax'' + By'' + C}.$$

This value might also have been deduced geometrically from the consideration that the ratio in which the line joining $x'y'$, $x''y''$ is cut, is equal to the ratio of the perpendiculars from these points upon the given line; but (Art. 31) these perpendiculars are

$$\frac{Ax' + By' + C}{\sqrt{(A^2 + B^2)}} \text{ and } \frac{Ax'' + By'' + C}{\sqrt{(A^2 + B^2)}}.$$

The negative sign in the preceding value arises from the fact that in the case of *internal* section to which the positive sign of $\frac{m}{n}$ corresponds (Art. 11) the perpendiculars fall on opposite sides of the given line, and must, therefore, be understood as having different signs (Art 31).

If a right line cut the sides of a triangle BC, CA, AB *in the points* LMN, *then*

$$\frac{BL \cdot CM \cdot AN}{CL \cdot AM \cdot BN} = -1.$$

Let the co-ordinates of the vertices be $x'y'$, $x''y''$, $x'''y'''$, then

$$\frac{BL}{CL} = -\frac{Ax'' + By'' + C}{Ax''' + By''' + C}; \quad \frac{CM}{AM} = -\frac{Ax''' + By''' + C}{Ax' + By' + C};$$

$$\frac{AN}{BN} = -\frac{Ax' + By' + C}{Ax'' + By'' + C};$$

and the truth of the theorem is manifest.

44. *To find the ratio in which the line joining two points* x_1y_1, x_2y_2, *is cut by the line joining two other points* x_3y_3, x_4y_4.

The equation of this latter line is (Art. 33)

$$(y_3 - y_4)\, x - (x_3 - x_4)\, y + x_3y_4 - x_4y_3 = 0.$$

Therefore by the last article

$$\frac{m}{n} = -\frac{(y_3 - y_4)\, x_1 - (x_3 - x_4)\, y_1 + x_3y_4 - x_4y_3}{(y_3 - y_4)\, x_2 - (x_3 - x_4)\, y_2 + x_3y_4 - x_4y_3}.$$

It is plain (by Art. 35) that this is the ratio of the two triangles whose vertices are

$$x_1y_1,\ x_3y_3,\ x_4y_4, \text{ and } x_2y_2,\ x_3y_3,\ x_4y_4,$$

as also is geometrically evident.

If the lines connecting any assumed point with the vertices of a triangle meet the opposite sides BC, CA, AB, *respectively in* L, M, N, then

$$\frac{BL \,.\, CM \,.\, AN}{CL \,.\, AM \,.\, BN} = +1.$$

Let the assumed point be x_4y_4, and the vertices x_1y_1, x_2y_2, x_3y_3, then

$$\frac{BL}{CL} = \frac{x_1(y_2 - y_4) + x_2(y_4 - y_1) + x_4(y_1 - y_2)}{x_1(y_4 - y_3) + x_4(y_3 - y_1) + x_3(y_1 - y_4)},$$

$$\frac{CM}{AM} = \frac{x_2(y_3 - y_4) + x_3(y_4 - y_2) + x_4(y_2 - y_3)}{x_1(y_2 - y_4) + x_2(y_4 - y_1) + x_4(y_1 - y_2)},$$

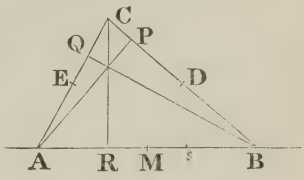
$$\frac{AN}{BN} = \frac{x_1(y_4 - y_3) + x_4(y_3 - y_1) + x_3(y_1 - y_4)}{x_2(y_3 - y_4) + x_3(y_4 - y_2) + x_4(y_2 - y_3)},$$

and the truth of the theorem is evident.

45. Analytic geometry adapts itself with peculiar readiness to the investigation of *loci*. We have only to find what relation the conditions of the question assign between the co-ordinates of the point whose locus we seek, and then the statement of this relation in algebraical language gives us at once the *equation* of the required locus.

Ex. 1. *Given base and difference of squares of sides of a triangle, to find the locus of vertex.*

Take the base for axis of x, and a perpendicular through its middle point for axis of y. Call the length of half base c, and let the co-ordinates of vertex be x, y. Then



$$BC^2 = CR^2 + RB^2 = y^2 + (c + x)^2.$$

In like manner $$AC^2 = y^2 + (c - x)^2;$$

therefore $$BC^2 - AC^2 = 4cx;$$

and, putting this equal to a constant,

$$4cx = m^2$$

is the equation of the locus of vertex; but this is (Art. 23) the equation of a line perpendicular to the base at a distance from middle point $= \frac{m^2}{4c}$, and, therefore (as easily appears), cutting the base so that the difference of the squares of segments = the difference of squares of sides (Euc. I. 47, Cor. 4).

Ex. 2. *Given base and sum of sides of a triangle, if the perpendicular be produced beyond the vertex until its whole length equal one of the sides, to find the locus of the extremity of the perpendicular.*

Take one extremity of the base (A) for origin; let the axis be the base and a perpendicular to it through A; and let us inquire what relation exists between the co-ordinates of the point whose locus we are seeking. The x of this point plainly is AR, and the y is, by hypothesis, = to AC; and if m be the given sum of sides,

$$BC = m - y.$$

Now (Euclid, II. 13),

$$BC^2 = AB^2 + AC^2 - 2AB \,.\, AR;$$ or, denoting AB by c,

$$(m - y)^2 = c^2 + y^2 - 2cx.$$

Reducing this equation, we get

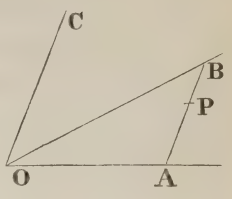
$$2my - 2cx = m^2 - c^2,$$

the equation of a right line.

Ex. 3. *Given two fixed lines,* OA *and* OB, *if any line be drawn*

to intersect them parallel to a third fixed line, OC, *to find the locus of the point where* AB *is cut in a given ratio.*

We may here employ oblique axes, since angles are not concerned (Art. 42). Let us take the fixed line OA for axis of x, and the fixed line OC for axis of y, then the equation of OB must be of the form $y = mx$, and it is required to find the locus of the point P cutting AB, so that AP may, for instance, $= \frac{1}{n}$ AB.



Since the point B lies on the line whose equation is $y = mx$, we have

$$AB = m\,OA,$$

therefore

$$AP = \frac{m}{n}\,OA,$$

but AP is the y of the point P, and OA its x, therefore the locus of P is expressed by the equation

$$y = \frac{m}{n}x,$$

and is, therefore, a right line through the point O.

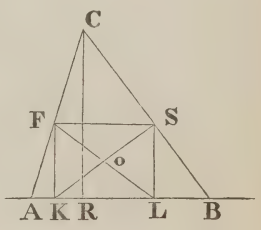
Ex. 3. *To find the locus of the middle points of rectangles inscribed in a given triangle.*

Let us take for axes CR and AB, as in Art. 42, Ex. 1. The equations of AC and BC are there given

$$\frac{y}{p} - \frac{x}{s'} = 1 \quad \text{and} \quad \frac{y}{p} + \frac{x}{s} = 1.$$

Now if we draw any line FS parallel to the base at a distance FK $= k$, and whose equation, therefore, is

$$y = k,$$



we can find the abscissæ of the points F and S, in which the line FS meets AC and BC, by substituting in the equations of AC and BC this value, $y = k$. Thus we get from the first equation

$$\frac{k}{p} - \frac{x}{s'} = 1 \;\therefore\; x \text{ or } KR = -s'\left(1 - \frac{k}{p}\right);$$

and from the second equation

$$\frac{k}{p} + \frac{x}{s} = 1 \;\therefore\; x \text{ or } RL = s\left(1 - \frac{k}{p}\right).$$

Having the abscissæ of F and S, we have (by Art. 11) the abscissa of the middle point of FS, viz., $x = \frac{s - s'}{2} \cdot \left(1 - \frac{k}{p}\right)$. This is evidently the abscissa of the middle point of the rectangle. But its ordinate is $y = \frac{k}{2}$. Now we want to find a relation which will subsist between this ordinate and abscissa whatever k be. We have only then to eliminate k between these equations, or substituting in the first the value of k $(= 2y)$ derived from the second, we have

$$2x = (s - s')\left(1 - \frac{2y}{p}\right),$$

or

$$\frac{2x}{s - s'} + \frac{2y}{p} = 1.$$

This is the equation of the locus which we seek. It obviously represents a right line, and if we examine the intercepts which it cuts off on the axes we shall find it to be the line joining the middle point of the perpendicular CR to the middle point of the base.

Ex. 4. *A line is drawn parallel to the base of a triangle, and its extremities joined transversely to the base, to find the locus of the point of intersection of the joining lines.*

The axes remaining as in the last example, let any parallel be drawn $(y = k)$; then one transverse line joins the points

B$(y = 0, x = s)$, and F$\left(y = k, x = -s' \cdot \overline{1 - \frac{k}{p}}\right)$ (see last Example), therefore its equation is

$$\left(s + s' \cdot \overline{1 - \frac{k}{p}}\right) y + kx - ks = 0.$$

In like manner, since AS joins

$$\text{A}(y = 0, x = -s') \text{ and } \text{S}\left(y = k, x = s \cdot \overline{1 - \frac{k}{p}}\right),$$

its equation is

$$\left(s' + s \cdot \overline{1 - \frac{k}{p}}\right) y - kx - ks' = 0.$$

Now since the point whose locus we are seeking lies on both the lines BF, AS, each of the equations just given expresses a relation which must be satisfied by the co-ordinates of the point O. Still, since these equations involve k, they express relations which

are only true for that particular point of the locus which corresponds to the case when the parallel FS is drawn at a height k above the base. If, however, between these two equations we eliminate the indeterminate k, we shall obtain a relation involving only the co-ordinates and known quantities, and which, since it must be satisfied whatever be the position of the parallel FS, will be the required equation of the locus.

Subtract the equations and the result becomes divisible by k. It is

$$\frac{s-s'}{p}y + 2x - \overline{s-s'} = 0, \text{ or } \frac{y}{p} + \frac{2x}{s-s'} = 1.$$

This is the equation of the locus we seek; but we have seen (Art. 42, Ex. 2) that this is the equation of MC, the bisector of the base.

Ex. 5. It may be useful to the student to see the same question solved with different axes of co-ordinates. And as this question involves no relations of angles, we may, without inconvenience, choose oblique axes.

Let us take for axes the sides of the triangle AC and CB; let their lengths be a and b; then any parallel to the base will cut off proportional intercepts μa and μb.

Hence the equations of the transversals will be

$$\frac{x}{a} + \frac{y}{\mu b} = 1, \text{ and } \frac{x}{\mu a} + \frac{y}{b} = 1.$$

Subtract these, and we get

$$\left(1 - \frac{1}{\mu}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 0;$$

or, dividing by the constant,

$$\frac{x}{a} - \frac{y}{b} = 0.$$

This relation, which must be always satisfied by the co-ordinates of the point of intersection, is, therefore, the equation of the locus we are seeking, but it is evidently the equation of a right line through the vertex of the triangle; and we shall have occasion to show in the next chapter (Art. 66), that this is the equation of the bisector of base, the sides of the triangle being taken for axes of co-ordinates.

Ex. 6. *A parallel is drawn to the base of a triangle, and perpendiculars to the sides erected at its extremities, find locus of their intersection.*

Take the same axes as in Ex. 4. Then the line FQ, which is a perpendicular to the line AC $\left(\frac{y}{p} - \frac{x}{s'} = 1\right)$, through the point $F\left(y = k,\ x = -s'.\overline{1 - \frac{k}{p}}\right)$, has for its equation

$$\frac{1}{p}\left(x + s'.\overline{1 - \frac{k}{p}}\right) + \frac{1}{s'}(y - k) = 0.$$

In like manner, the equation of SQ is

$$\frac{1}{p}\left(x - s.\overline{1 - \frac{k}{p}}\right) - \frac{1}{s}(y - k) = 0.$$

We have to eliminate k between these equations. Put them into the form

FQ $\qquad \frac{x}{p} + \frac{s'}{p} + \frac{y}{s'} = k\left(\frac{1}{s'} + \frac{s'}{p^2}\right)$, and

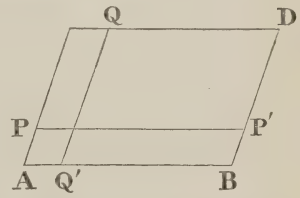
SQ $\qquad \frac{y}{s} - \frac{x}{p} + \frac{s}{p} = k\left(\frac{1}{s} + \frac{s}{p^2}\right)$.

Eliminate k between these equations, and we have, for the equation of the locus,

$$\left(\frac{1}{s} + \frac{s}{p^2}\right)\left(\frac{x}{p} + \frac{s'}{p} + \frac{y}{s'}\right) = \left(\frac{1}{s'} + \frac{s'}{p^2}\right)\left(\frac{y}{s} - \frac{x}{p} + \frac{s}{p}\right);$$

but this is evidently the equation of a right line, since x and y are only in the first degree, and it will be found that it passes through the vertex of the given triangle, for the co-ordinates of the vertex $x = 0$, $y = p$, will satisfy the equation.

Ex. 7. PP′ *and* QQ′ *are any two parallels to the sides of a parallelogram; to find the locus of the intersection of the lines* PQ *and* P′Q′.



Let us take two of the sides for our axes, and let the lengths of the sides be a and b, and let $AQ' = m$, $AP = n$. Then the equation of PQ, joining P $(y = n,\ x = 0)$ to Q $(x = m,\ y = b)$ is

$$(b - n)\,x - my + mn = 0,$$

and the equation of P′Q′ joining P′($x = a, y = n$), to Q′($x = m, y = 0$) is

$$nx - (a - m)\,y - mn = 0.$$

There being *two* indeterminates, m and n, we should at first suppose that it would not be possible to eliminate them from *two* equations. However, if we add the above equations, it will be found that both vanish together, and we get for our locus

$$bx - ay = 0,$$

the equation of the diagonal of the parallelogram.

Ex. 8. Example 4 is only a particular case of the following: *A line is drawn parallel to the ~~sides~~ base of a triangle, and the points where it meets sides joined to* any *two fixed points on the base; to find the locus of the point of intersection of the joining lines.*

We shall preserve the same axes, &c., as in Ex. 4, and let the co-ordinates of the fixed points, T and V, on the base, be for T $y = 0$, $x = m$, and for V $y = 0$, $x = n$.

We already found (Ex. 4) the co-ordinates of

$$\mathrm{F}\left(y = k,\ x = -s'.\sqrt{1 - \frac{k}{p}}\right),\quad \mathrm{S}\left(y = k,\ x = s.\sqrt{1 - \frac{k}{p}}\right).$$

Hence, if we form the equation of FT, we get

$$\left(s'.\sqrt{1 - \frac{k}{p}} + m\right)y + kx - km = 0,$$

and equation of SV is

$$\left(s.\sqrt{1 - \frac{k}{p}} - n\right)y - kx + kn = 0.$$

We can eliminate k by putting the equations into the form

$$\text{FT}\qquad (s' + m)y - k\left(\frac{s'}{p}y - x + m\right) = 0,$$

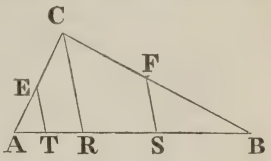
$$\text{and SV}\qquad (s - n)\,y - k\left(\frac{s}{p}y + x - n\right) = 0.$$

Hence, eliminating k, we get for the equation of the locus

$$(s - n)\left(\frac{s'}{p}y - x + m\right) = (s' + m)\left(\frac{s}{p}y + x - n\right).$$

But this is the equation of a right line, since x and y are only in the first degree.

Ex. 9. *If on the base of a triangle we take any portion* AT, *and on the other side of the base another portion* BS, *in a fixed ratio to* AT, *and erect the perpendiculars** ET *and* FS, *to find the locus of* O, *the point of intersection of* EB *and* FA.



Call AT k, let the fixed ratio be m, then BS will $= mk$; the co-ordinates of S will be $y = 0$, $x = s - mk$, and of T, $y = 0$, $x = -(s' - k)$.

The ordinates of E and F will be found by substituting these values of x in the equations of AC and BC. We get for

$$\text{E}, \quad x = -(s' - k), \quad y = \frac{pk}{s'},$$

and for

$$\text{F}, \quad x = s - mk, \quad y = \frac{mpk}{s}.$$

Now form the equations of the transverse lines, and the equation of EB is

$$(s + s' - k)y + \frac{pk}{s'}x - \frac{pks}{s'} = 0,$$

and the equation of AF is

$$(s + s' - mk)\,y - \frac{mpk}{s}x - \frac{mpks'}{s} = 0.$$

To eliminate k subtract the equations, and they will become divisible by k, and we get

$$(m - 1)y + \left(\frac{mp}{s} + \frac{p}{s'}\right)x + \left(\frac{mps'}{s} - \frac{ps}{s'}\right) = 0,$$

which is the equation of a right line.

Ex. 10. *Given a point and two fixed lines: draw any two lines through the fixed point, and join transversely the points where they meet the fixed lines, to find the locus of intersection of the transverse lines.*

Take the fixed lines for axes, and let the equations of the lines through the fixed point be

$$\frac{x}{m} + \frac{y}{n} = 1, \quad \text{and} \quad \frac{x}{m'} + \frac{y}{n'} = 1.$$

* The following investigation will apply equally if ET and FS be not perpendicular, but parallel to any fixed line CR.

The condition that these lines should pass through the fixed point $x'y'$ gives us

$$\frac{x'}{m} + \frac{y'}{n} = 1, \quad \text{and } \frac{x'}{m'} + \frac{y'}{n'} = 1;$$

or, subtracting,

$$x'\left(\frac{1}{m} - \frac{1}{m'}\right) + y'\left(\frac{1}{n} - \frac{1}{n'}\right) = 0.$$

Now the equations of the transverse lines clearly are

$$\frac{x}{m} + \frac{y}{n'} = 1, \quad \text{and } \frac{x}{m'} + \frac{y}{n} = 1;$$

or, subtracting,

$$x\left(\frac{1}{m} - \frac{1}{m'}\right) - y\left(\frac{1}{n} - \frac{1}{n'}\right) = 0.$$

Now from this and the equation just found we can eliminate

$$\left(\frac{1}{m} - \frac{1}{m'}\right) \text{ and } \left(\frac{1}{n} - \frac{1}{n'}\right),$$

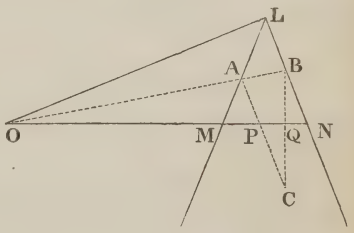
and we have

$$x'y + y'x = 0,$$

the equation of a right line through the origin.

Ex. 11. *Two vertices of a triangle* ABC *move on fixed right lines* LM, LN, *and the three sides pass through three fixed points* O, P, Q *which lie on a right line; find the locus of the third vertex.*

Take for axis of x the right line OP, containing the three fixed points, and for axis of y the line OL joining the intersection of the two fixed lines to the point O through which the base passes. Let



$$\text{OL} = b, \quad \text{OM} = a, \quad \text{ON} = a', \quad \text{OP} = c, \quad \text{OQ} = c'.$$

Then obviously the equations of LM, LN are

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ and } \frac{x}{a'} + \frac{y}{b} = 1.$$

Now in solving loci we have generally our choice of two methods: we may either, as in previous examples, introduce an indeterminate, which the conditions of the problem enable us afterwards to eliminate; or else we may endeavour to express directly the conditions of the problem in terms of the co-ordinates of the

point whose locus we are seeking, and thus obtain a relation between these co-ordinates, which is the equation of the required locus. Thus in the present example we might, by the first method, assume for the equation of OB, $y = kx$, where k is indeterminate, thence obtain the co-ordinates of A and B, thence form the equations of AP and BQ, and finally, by eliminating k from these, find the equation of the locus of C.

Or else we may begin by assuming the co-ordinates of C $x'y'$; thence form the equations of CP, CQ; thence obtain the co-ordinates of A and B; and finally, by expressing the condition that the line joining AB shall pass through O, have a relation between $x'y'$, which is the equation of the required locus.

We shall now for variety adopt the latter method. The equation of CP through $x'y'$ and P ($y = 0$, $x = c$) is

$$(x' - c)\, y - y'x + cy' = 0.$$

The co-ordinates of A, the intersection of this line with

$$\frac{x}{a} + \frac{y}{b} = 1,$$

are

$$x_1 = \frac{ab\,(x' - c) + acy'}{b\,(x' - c) + ay'}; \quad y_1 = \frac{b\,(a - c)\,y'}{b\,(x' - c) + ay'}.$$

The co-ordinates of B are found by simply accentuating the letters in the preceding:

$$x_2 = \frac{a'b\,(x' - c') + a'c'y'}{b\,(x' - c') + a'y'}; \quad y_2 = \frac{b\,(a' - c')\,y'}{b\,(x' - c') + a'y'}.$$

Now the condition that two points, x_1y_1, x_2y_2, shall lie on a right line passing through the origin, is simply (Art. 35) $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.

Applying this condition we have

$$\frac{b\,(a - c)\,y'}{ab\,(x' - c) + acy'} = \frac{b\,(a' - c')\,y'}{a'b\,(x' - c') + a'c'y'}.$$

This being a relation then which must always be satisfied by the co-ordinates $x'y'$, the equation of the locus is

$$(a - c)\,[a'b\,(x - c') + a'c'y] = (a' - c')\,[ab\,(x - c) + acy],$$

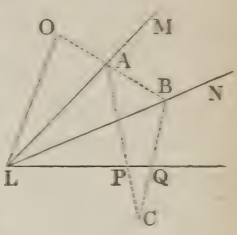
or

$$\frac{(ac' - a'c)\,x}{cc'\,(a - a') - aa'\,(c - c')} + \frac{y}{b} = 1,$$

the equation of a right line through the point L.

Ex. 12. *If in the last example the points* P, Q, *lie on a right line passing not through* O *but through* L, *find the locus of vertex.*

Take for axis of x the line LP, and for axis of y the line LO. Let LP $= a$, LQ $= a'$, LO $= b$, and let the equations of LM, LN be $y = mx$ and $y = m'x$. I adopt the method of solution used in the last example. The equation of CP through $x'y'$ and $(x = a,\ y = 0)$ is



$$y'(x - a) = (x' - a)\,y.$$

The co-ordinates therefore of the point A where this line meets $y = mx$ are

$$x_1 = \frac{ay'}{y' - mx' + am}, \quad y_1 = \frac{amy'}{y' - mx' + am}$$

In like manner the co-ordinates of B are

$$x_2 = \frac{a'y'}{y' - m'x' + a'm'}, \quad y_2 = \frac{a'm'y'}{y' - m'x' + a'm'}.$$

We must now express that the line joining these points passes through O. But (Art. 33) the intercept on the axis of y by the line joining two points x_1y_1, x_2y_2, is $\frac{x_1y_2 - x_2y_1}{x_1 - x_2}$; we must therefore form this quantity and equate it to b. This gives

$$x_1(y_2 - b) = x_2(y_1 - b).$$

Substitute in this the values just obtained for x_1y_1, x_2y_2, and clear of fractions; the equation becomes divisible by y', and we have for the relation to be satisfied by the point $x'y'$,

$$a\{(a'm' - b)\,y + m'b\,(x - a)\} = a'\{(am - b)\,y + mb\,(x - a)\},$$

the equation of a right line.

Ex. 13. *Given bases and sum of areas of any number of triangles having a common vertex, to find its locus.*

Let the equations of the bases be

$$x\cos\alpha + y\sin\alpha - p = 0, \quad x\cos\beta + y\sin\beta - p_1 = 0,$$
$$x\cos\gamma + y\sin\gamma - p_2 = 0, \text{ \&c.}$$

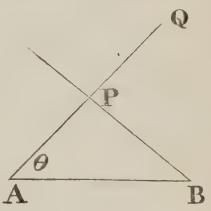
and their lengths, a, b, c, &c.; and let the given sum $= m^2$; then, since (Art. 31) $x\cos\alpha + y\sin\alpha - p$ denotes the perpendicular from the point xy on the first line, $a(x\cos\alpha + y\sin\alpha - p)$ will be double the area of the first triangle, and the equation of the locus will be

$$a(x\cos\alpha + y\sin\alpha - p) + b(x\cos\beta + y\sin\beta - p_1) + c(x\cos\gamma + y\sin\gamma - p_2) + \&c. = 2m^2,$$

which, since it contains x and y only in the first degree, will represent a right line.

46. We shall conclude this chapter by showing how questions may be solved by the help of *polar* co-ordinates (see Art. 40). It is, in general, convenient to use this method, if the question be to find the locus of the extremities of lines drawn through a fixed point according to any given law. For example,

Ex. 1. A *and* B *are two fixed points; draw through* B *any line, and let fall on it a perpendicular from* A, AP; *produce* AP *so that the rectangle* AP . AQ *may be constant: to find the locus of the point* Q.

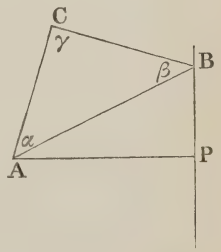


Take A for the pole, and AB for the fixed axis, then AQ is our radius vector, designated by ρ, and the angle $QAB = \theta$, and our object is to find the relation existing between ρ and θ. Let us call the constant length $AB = c$, and from the right-angled triangle APB we have $AP = c\cos\theta$, but $AP \cdot AQ = \text{const} = k^2$, therefore,

$$\rho c\cos\theta = k^2, \text{ or } \rho\cos\theta = \frac{k^2}{c};$$

but we have seen (Art. 40) that this is the equation of a right line perpendicular to AB, and at a distance from $A = \frac{k^2}{c}$.

Ex. 2. *Given the angles of a triangle; one vertex* A *is fixed, another* B *moves along a fixed right line, to find the locus of the third.*



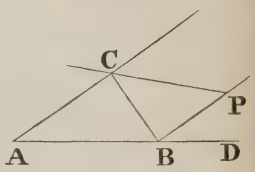
Take the fixed vertex A for pole, and AP perpendicular to the fixed line for axis, then $AC = \rho$, $CAP = \theta$. Now since the angles of ABC are given, AB is in a fixed ratio to AC ($= m$AC) and $BAP = \theta - a$; but $AB \cdot \cos BAP = AP$; therefore, if we call AP, a, we have

$$m\rho\cos(\theta - a) = a,$$

which (by Art. 40) is the equation of a right line, making an angle α with the given line, and at a distance from $A = \frac{a}{m}$.

Ex. 3. *Given base and sum of sides of a triangle, if at either extremity of the base* B *a perpendicular be erected to the conterminous side* BC, *to meet the external bisector of vertical angle* CP, *to find the locus of the point of meeting* P.

Let us take the point B for our pole, then BP will be our radius vector ρ; and let us take the base produced for our fixed axis, then $PBD = \theta$, and our object is to express ρ in terms of θ. Let us designate the sides and opposite angles of the triangle a, b, c, A, B, C, then it is easy to see, that the angle $BCP = 90° - \frac{1}{2}C$, and from the triangle PCB, that $a = \rho \tan \frac{1}{2}C$. Hence it is evident, that if we could express a and $\tan \frac{1}{2}C$ in terms of θ, we could express ρ in terms of θ. Now from the given triangle we have



$$b^2 = a^2 + c^2 - 2ac \cos B,$$

but if the given sum of sides be m, we may substitute for b, $m - a$; and cos B plainly = sin θ; hence

$$m^2 - 2am + a^2 = a^2 + c^2 - 2ac \sin \theta,$$

hence

$$a = \frac{m^2 - c^2}{2(m - c \sin \theta)}.$$

Thus we have expressed a in terms of θ and constants, and it only remains to find an expression for $\tan \frac{1}{2}C$.

But

$$\tan^2 \tfrac{1}{2}C = \frac{(s-a)(s-b)}{s(s-c)}, \text{ (\textit{Luby's Trig.} § 41),}$$

$$= \frac{c^2 - (a-b)^2}{(a+b+c)(a+b-c)}, \text{ or (since } a+b=m) = \frac{c^2 - (a-b)^2}{m^2 - c^2};$$

again

$$a - b = 2a - m = c\,\frac{m \sin \theta - c}{m - c \sin \theta};$$

hence

$$c^2 - (a-b)^2 = c^2\,\frac{(m^2 - c^2)\cos^2\theta}{(m - c \sin \theta)^2},$$

therefore

$$\tan \tfrac{1}{2}C = \frac{c \cos \theta}{m - c \sin \theta}.$$

We are now able to express ρ in terms of θ, for, substitute in

the equation $a = \rho \tan \frac{1}{2}C$ the values we have found for a and $\tan \frac{1}{2}C$, and we get

$$\frac{m^2 - c^2}{2(m - c\sin\theta)} = \frac{\rho c \cos\theta}{(m - c\sin\theta)}, \quad \text{or } \rho\cos\theta = \frac{m^2 - c^2}{2c}.$$

Hence the locus is a line perpendicular to the base of the triangle at a distance from $B = \frac{m^2 - c^2}{2c}$.

The student may exercise himself with the corresponding locus, if CP had been the *internal* bisector, and if the *difference* of sides had been given.

Ex. 4. *Given n fixed right lines and a fixed point* O; *if through this point any radius vector be drawn meeting the right lines in the points* $r_1, r_2, r_3 \ldots r_n$, *and on this a point* R *be taken such that* $\frac{n}{OR} = \frac{1}{Or_1} + \frac{1}{Or_2} + \frac{1}{Or_3} + \ldots \frac{1}{Or_n}$, *to find the locus of* R.

Let the equations of the right lines be

$$\rho\cos(\theta - a) = p_1; \quad \rho\cos(\theta - \beta) = p_2, \text{ \&c.}$$

Hence it is easy to see that the equation of the locus is

$$\frac{n}{\rho} = \frac{\cos(\theta - a)}{p_1} + \frac{\cos(\theta - \beta)}{p_2} + \text{\&c.}$$

the equation of a right line (Art. 40). This theorem is only a particular case of a general one which we shall prove afterwards.

CHAPTER IV.

APPLICATION OF ABRIDGED NOTATION TO THE EQUATION OF THE RIGHT LINE.

47. In the preceding examples we generally endeavoured to simplify our calculations by a particular choice of the axes of co-ordinates. It will in many cases, however, be found more advantageous to give the axes their most general position; the equations gaining in symmetry more than an equivalent for what they seem to lose in simplicity. And in the present chapter we proceed to

lay down some principles of importance in the geometrical interpretation of equations, whatever be the position of the axes.

It may be well, however, first to remark, that no transformation of co-ordinates can increase or diminish the *degree* of an equation. It cannot *increase* the degree, for (Art. 14) in the most general case of transformation we have to substitute for x and y functions of the form $(Ax + By + C)$. If, therefore, the highest powers in the original equation be x^m, y^m, &c., the highest terms in the transformed equation will be $(Ax + By + C)^m$, &c., which evidently cannot contain powers of x or y beyond the m^{th} degree.

Nor can transformation *diminish* the degree of an equation, since by transforming the transformed equation back again to the old axes, we must fall back on the original equation, and if the first transformation had diminished the degree of the equation, the second should increase it, contrary to what has been just proved.

48. Let the equation of one line be $x \cos \alpha + y \sin \alpha - p = 0$, and of another $x \cos \beta + y \sin \beta - p' = 0$, then the equation

$$(x \cos \alpha + y \sin \alpha - p) - k (x \cos \beta + y \sin \beta - p') = 0,$$

where k is any constant, will represent a right line passing through the intersection of these two lines; for it is evidently the equation of a right line, and since the co-ordinates of the point of intersection must satisfy the two equations

$$x \cos \alpha + y \sin \alpha - p = 0, \text{ and } x \cos \beta + y \sin \beta - p' = 0,$$

they must satisfy the equation

$$(x \cos \alpha + y \sin \alpha - p) - k (x \cos \beta + y \sin \beta - p') = 0,$$

for, when substituted in it, they will render each member = 0.

The principle here established is of great importance, and one of which we shall make constant use. It may be generalized as follows: Let $s = 0$, $s' = 0$, be the abbreviations for two equations of any degree; then $s + ks' = 0$ denotes a curve passing through every point common to the loci represented by $s = 0$, $s' = 0$. For the equation is manifestly satisfied by any values of the co-ordinates which make s and s' separately = 0.

49. Attention to this principle often enables us simply to construct geometrically the right line represented by a given equa-

tion. For the very form of the equation often indicates in this manner some remarkable point through which it passes, and if we can thus determine two points in the right line, we can of course construct the right line. The learner will find it a profitable exercise to go over the examples of the last chapter, and to endeavour in this manner to construct geometrically the locus found in each. Thus in Ex. 6, p. 39, we obtained the equation

$$\left(\frac{1}{s}+\frac{s}{p^2}\right)\left(\frac{x}{p}+\frac{y}{s'}+\frac{s'}{p}\right)=\left(\frac{1}{s'}+\frac{s'}{p^2}\right)\left(\frac{y}{s}-\frac{x}{p}+\frac{s}{p}\right).$$

Now, from what has been said, it appears that this is a line passing through the intersection of the lines

$$\frac{x}{p}+\frac{y}{s'}+\frac{s'}{p}=0, \quad \text{and } \frac{y}{s}-\frac{x}{p}+\frac{s}{p}=0.$$

But on turning to the example cited, it will be found that these latter are the equations of lines drawn through the extremities of the base of the triangle perpendicular to the conterminous sides; the locus is, therefore, the right line joining the intersection of these perpendiculars to the vertex. And so in like manner for the other examples.

Or, again, the same principle will often save us labour in the formation of equations. Thus, if it were required to find the equation of the line joining the point $x'y'$ to the intersection of the right lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$, instead of first finding the co-ordinates of the intersection of these lines, and then forming the required equation by Art. 33, we may at once write down the equation of a line through the intersection of the given lines

$$Ax + By + C - k(A'x + B'y + C') = 0,$$

and determine k from the consideration that this equation must be satisfied by the co-ordinates $x'y'$. The required equation is, therefore,

$$(A'x' + B'y' + C')(Ax + By + C) = (Ax' + By' + C)(A'x + B'y + C').$$

50. The reader will often find it convenient to use abbreviations for the quantities $x \cos \alpha + y \sin \alpha - p$, &c. Let us call $x \cos \alpha + y \sin \alpha - p$, α, and $x \cos \beta + y \sin \beta - p'$, β, then we have proved that, if $\alpha = 0$, $\beta = 0$, be the equations of two lines,

$\alpha - k\beta = 0$ is the equation of a line passing through the point of intersection of these two lines. We shall, for brevity, call these the lines α, β, and their point of intersection the point $\alpha\beta$. We shall too have occasion often to use abbreviations for the equations of lines in the form $Ax + By + C = 0$. We shall in these cases make use of Roman letters, reserving the letters of the Greek alphabet to intimate that the equation is in the form

$$x \cos \alpha + y \sin \alpha - p = 0.$$

51. The principle of Art. 48 may be otherwise stated thus: *Three right lines will pass through the same point, if, their equations being multiplied each by any constant quantity and added together, the sum is identically* = 0. For, if between three equations $\alpha = 0$, $\beta = 0$, $\gamma = 0$, there exist the identical relation $l\alpha + m\beta + n\gamma = 0$, then the equation of the third line is $\gamma = -\frac{l}{n}\alpha - \frac{m}{n}\beta = 0$, an equation of the form discussed in Art. 48; or, more directly, it is evident that any values of the co-ordinates, which make $\alpha = 0$, $\beta = 0$, must, by virtue of the relation, $l\alpha + m\beta + n\gamma = 0$, make $\gamma = 0$ also.

52. Let us take, as an illustration of the application of this principle, the first example, Art. 42, "the three perpendiculars of a triangle meet in a point."

Let the co-ordinates of the vertices of the triangle be $x_1 y_1$, $x_2 y_2$, $x_3 y_3$. Now (Arts. 33, 39) the equation of a line through $x_3 y_3$, perpendicular to the line joining $x_1 y_1$, $x_2 y_2$, is

$$(x_1 - x_2)(x - x_3) + (y_1 - y_2)(y - y_3) = 0.$$

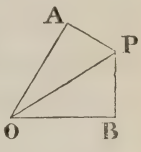
The advantage of using symmetrical equations is, that no new calculation is necessary to find the equations of the other two perpendiculars, but that we have merely to interchange $x_1 x_2 x_3$, &c., as at note, p. 22. The equations are

$$(x_2 - x_3)(x - x_1) + (y_2 - y_3)(y - y_1) = 0,$$
$$(x_3 - x_1)(x - x_2) + (y_3 - y_1)(y - y_2) = 0.$$

Now, it will be seen that when these three equations are added together, their sum vanishes; therefore, by the principle just laid down, the three lines meet in a point. The reader will find it useful to work out by symmetrical equations the other examples

of Art. 42. We shall presently give still shorter solutions of the same questions.

53. We proceed to examine the meaning of the coefficient k in the equation $\alpha - k\beta = 0$. We saw (Art. 31) that the quantity α (that is, $x \cos \alpha + y \sin \alpha - p$) denoted the length of the perpendicular let fall from any point xy, on the line OA (which we suppose represented by α). Similarly, that β is the length of the perpendicular from the point xy, on the line OB, represented by β. Hence the equation



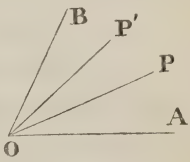
$$\alpha - k\beta = 0, \qquad \left(\text{or } \frac{\alpha}{\beta} = k\right),$$

asserts, that if from any point of the locus represented by it, perpendiculars be let fall on the lines OA, OB, the ratio of these perpendiculars, that is, $\frac{PA}{PB}$, will be constant, and $= k$. Hence the locus represented by $\alpha - k\beta = 0$ is a right line through O, and

$$k = \frac{PA}{PB}, \qquad \text{or} = \frac{\sin POA}{\sin POB}.$$

It follows from the conventions concerning signs (Art. 31) that $\alpha + k\beta = 0$, denotes a right line dividing *externally* the angle AOB into parts such that $\frac{\sin POA}{\sin POB} = k$.

54. If $\alpha - k\beta = 0$, $\alpha - k'\beta = 0$, be the equations of two lines, then $\frac{k}{k'}$ will be the *anharmonic relation* of the pencil formed by the four lines α, β, $\alpha - k\beta$, $\alpha - k'\beta$, for



$$k = \frac{\sin POA}{\sin POB}, \qquad k' = \frac{\sin P'OA}{\sin P'OB},$$

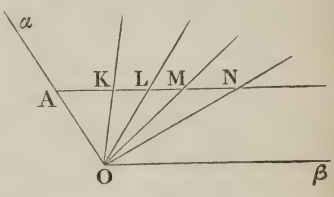
$$\therefore \frac{k}{k'} = \frac{\sin POA \,.\, \sin P'OB}{\sin POB \,.\, \sin P'OA};$$

but this is the anharmonic ratio of the pencil.*

* I suppose the reader to be already acquainted with the fundamental theorem, that if these four lines be met by any transverse line in four points A, B, P, P′: then the ratio $\frac{AP \,.\, PB}{BP \,.\, P'A}$ is constant, and $= \frac{k}{k'}$. In fact, let the perpendicular from O on the transverse

The pencil is an *harmonic* pencil when $\frac{k}{k'} = -1$, for then the angle AOB is divided internally and externally into parts whose sines are in the same ratio. Hence we have the important theorem, *two lines whose equations are* $\alpha - k\beta = 0$, $\alpha + k\beta = 0$, *form with* α, β, *an harmonic pencil.*

55. In general the anharmonic relation of four lines $\alpha - k\beta$, $\alpha - l\beta$, $\alpha - m\beta$, $\alpha - n\beta$, is $\frac{(n-l)(m-k)}{(n-m)(l-k)}$. For let the pencil be cut by any parallel to β in the four points K, L, M, N and the ratio is $\frac{\text{NL . MK}}{\text{NM . LK}}$. But since β has the same value for each of these four points, the perpendiculars from these points on α are (by virtue of the equations of the lines) proportional to k, l, m, n; and AK, AL, AM, AN, are evidently proportional to these perpendiculars; hence NL is proportional to $n - l$; MK to $m - k$; NM to $n - m$; and LK to $l - k$.



56. We infer, as a particular case of Art. 53, that *the equation of the line bisecting the angle between two lines* α *and* β *is* $\alpha - \beta = 0$; for the perpendiculars on the two lines from any point of the bisector are equal. That is, writing at full length, the equation of the line bisecting the angle between the lines

$$x \cos\alpha + y \sin\alpha - p = 0, \text{ and } x \cos\beta + y \sin\beta - p' = 0,$$

is
$$x \cos\alpha + y \sin\alpha - p = x \cos\beta + y \sin\beta - p'.$$

If we were required to find the equation of the line bisecting the angle between the lines

$$Ax + By + C = 0 \text{ and } A'x + B'y + C' = 0,$$

we should (Art. 30) reduce each equation to the form

$$x \cos\alpha + y \sin\alpha - p = 0,$$

and we should find for the equation of the bisector

$$\frac{Ax + By + C}{\sqrt{(A^2 + B^2)}} = \frac{A'x + B'y + C'}{\sqrt{(A'^2 + B'^2)}}.$$

---

line $= p$, then p . AP = OA . OP . sin POA (both being double the area POA); p . P'B = OP' . OB . sin P'OB; p . BP = OB . OP sin BOP; p . P'A = OP' . OA sin P'OA; hence the truth of the theorem can immediately be deduced.

The equation of the *external* bisector is, in like manner, $\alpha + \beta = 0$, and we see (from Art. 54) that the four lines α, β, $\alpha - \beta$, $\alpha + \beta$, form an harmonic pencil as is geometrically evident.

57. *The three internal bisectors of the angles of a triangle meet in a point.*

Let the sides be α, β, γ, and the equations of the bisectors are $\alpha - \beta = 0$, $\beta - \gamma = 0$, $\gamma - \alpha = 0$, and since the sum of these equations is identically $= 0$, it follows from Art. 51, that the three lines meet in a point.

Any two external bisectors of the angles of a triangle meet on the internal bisector of the third.

For their equations are $\alpha + \beta = 0$, $\beta + \gamma = 0$, $\gamma - \alpha = 0$; and if we multiply the second by -1, and add the three together, the sum is identically $= 0$.

58. The lines $\alpha - k\beta = 0$, and $k\alpha - \beta = 0$, are plainly such that one makes the same angle with the line α which the other makes with the line β, and are therefore equally inclined to the bisector $\alpha - \beta$.

Hence, *if through the vertices of a triangle there be drawn any three lines meeting in a point, the three lines drawn through the same angles, equally inclined to the bisectors of the angles, will also meet in a point.*

Let the sides of the triangle be α, β, γ, and let the equations of the first three lines be

$$l\alpha - m\beta = 0, \quad m\beta - n\gamma = 0, \quad n\gamma - l\alpha = 0,$$

which, by the principle of Art. 51, are the equations of three lines meeting in a point, and which obviously pass through the points $\alpha\beta$, $\beta\gamma$, and $\gamma\alpha$. Now, from this Article, the equations of the second three lines will be

$$\frac{\alpha}{l} - \frac{\beta}{m} = 0, \quad \frac{\beta}{m} - \frac{\gamma}{n} = 0, \quad \text{and } \frac{\gamma}{n} - \frac{\alpha}{l} = 0,$$

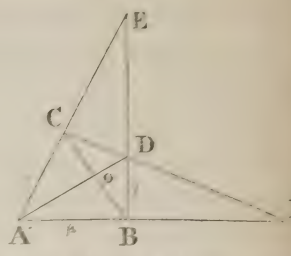
which (by Art. 51) must also meet in a point.

59. We may apply these principles to deduce analytically the harmonic properties of a complete quadrilateral.

Let the equation of AC be $\alpha = 0$; of AB, $\beta = 0$; of BD, $\gamma = 0$; of AD, $l\alpha - m\beta = 0$; and of BC, $m\beta - n\gamma = 0$. Then we are able

to express in terms of these quantities the equations of all the other lines of the figure.

For instance, the equation of CD is $$l\alpha - m\beta + n\gamma = 0,$$ for it is the equation of a right line passing through the intersection of $l\alpha - m\beta$ and γ, that is, the point D, and of α and $m\beta - n\gamma$, that is, the point C. Again, $l\alpha - n\gamma = 0$ is the equation of OE, for it evidently passes through the intersection of α and γ or E, and it also passes through the intersection of AD and BC, since it is $= (l\alpha - m\beta) + (m\beta - n\gamma)$.



The equation of EF is

$$l\alpha + n\gamma = 0.$$

This is evidently the equation of a right line passing through the point E or $\alpha\gamma$, and it also passes through the point F, for it is $= (l\alpha - m\beta + n\gamma) + m\beta$, and, therefore, passes through the intersection of CD and AB.

From Art. 54 it appears, that the four lines EA, EO, EB, and EF, form an harmonic pencil, for their equations have been shown to be $$\alpha,\ \gamma,\ l\alpha - n\gamma,\ \text{and}\ l\alpha + n\gamma.$$

Again, the equation of FO is

$$l\alpha - 2m\beta + n\gamma = 0,$$

for this is a line passing through the points $(l\alpha - n\gamma,\ \beta)$, and $(l\alpha - m\beta,\ m\beta - n\gamma)$.

Hence (Art. 54) the four lines FE, FC, FO, and FB, are an harmonic pencil, for their equations are

$$l\alpha - m\beta + n\gamma,\ \beta,\ \ l\alpha - m\beta + n\gamma - m\beta,\ \text{and}\ l\alpha - m\beta + n\gamma - m\beta.$$

Again, OC, OE, OD, OF, are an harmonic pencil, for their equations are

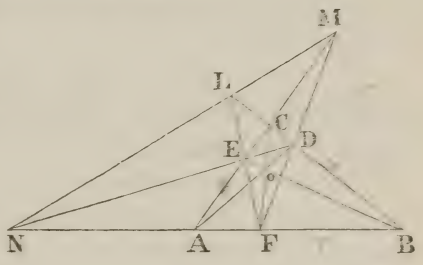
$$l\alpha - m\beta,\ m\beta - n\gamma,\ l\alpha - m\beta + m\beta - n\gamma,\ \text{and}\ l\alpha - m\beta - (m\beta - n\gamma).$$

60. We shall choose, as another example, the system of lines formed by drawing through the angles of a triangle three lines meeting in a point.

Let the equation of AB be $\gamma = 0$; of AC $\beta = 0$; of BC $\alpha = 0$;

then we shall assume for OC $l\alpha - m\beta$; for OA $m\beta - n\gamma$; and for OB $n\gamma - l\alpha$ (as in Art. 58); these three lines meet in a point, since these three quantities added together are = 0.

Now we can form the equations of all the other lines in the figure.



For example, the equation of EF is

$$m\beta + n\gamma - l\alpha = 0,$$

since it passes through the points $(\beta, n\gamma - l\alpha)$ or E, and $(\gamma, m\beta - l\alpha)$ or F.

In like manner, the equation of DF is

$$l\alpha - m\beta + n\gamma = 0,$$

and of DE

$$l\alpha + m\beta - n\gamma = 0.$$

Now we can prove, that the three points L, M, N are all in one right line, whose equation is

$$l\alpha + m\beta + n\gamma = 0,$$

for this line passes through the points $(l\alpha + m\beta - n\gamma, \gamma)$ or N, $(l\alpha - m\beta + n\gamma, \beta)$ or M, and $(m\beta + n\gamma - l\alpha, \alpha)$ or L.

The equations of CN is

$$l\alpha + m\beta = 0,$$

for this is evidently a line through (α, β) or C, and it also passes through N, since it = $(l\alpha + m\beta + n\gamma) - n\gamma$.

Hence BN is cut harmonically, for the equations of the four lines CN, CA, CF, CB, are,

$$\alpha, \beta, l\alpha - m\beta, l\alpha + m\beta.$$

We shall often afterwards meet with equations of the form discussed in this article.

61. *If two triangles be such that the intersections of the corresponding sides lie on the same right line, the lines joining the corresponding vertices meet in a point.*

Let the sides of the triangles be α, β, γ; α', β', γ'; then the equation of the line (L) on which the sides intersect must admit of being expressed in any of the three forms

$$L = a\alpha + a'\alpha' = b\beta + b'\beta' = c\gamma + c'\gamma' = 0.$$

Now $a\alpha - b\beta = 0$, is the equation of the line joining $a\beta$, $a'\beta'$; for it obviously passes through $a\beta$, and it follows from the equivalence of the first two expressions for L that $a\alpha - b\beta = b'\beta' - a'\alpha'$: it must therefore also denote a line passing through $a'\beta'$. But the three lines

$$a\alpha - b\beta = 0, \quad b\beta - c\gamma = 0, \quad c\gamma - a\alpha = 0,$$

meet in a point. It is easy to show that the converse of this theorem is equally true.

The three points, where the line joining $a\beta'$, $a'\beta$, *meets* γ, *where the line joining* $\beta\gamma'$, $\beta'\gamma$, *meets* α, *and where the line joining* $\gamma a'$, $a\gamma'$, *meets* β, *lie in one right line.*

We have as before

$$a\alpha - b'\beta' = b\beta - a'\alpha',$$

therefore $a\alpha - b'\beta'$ $(= a\alpha + b\beta - \mathrm{L}) = 0$, is the equation of the line joining $a'\beta$, $a\beta'$. And $a\alpha + b\beta + c\gamma - \mathrm{L} = 0$, is the equation of a line through the point where this line meets γ. But the symmetry of the equation shows that this line also passes through the point where the line joining $\beta\gamma'$, $\beta'\gamma$, meets α, and where the line joining $\gamma a'$, $a\gamma'$, meets β.*

62. In the last Articles we were given the equations of three lines α, β, γ, and we were able to express all the other lines with which we were concerned, by equations of the form

$$\alpha + k\beta + l\gamma = 0.$$

We proceed now to show that there is *no line* whose equation may not be put into the same form, for since this equation contains two arbitrary constants, k and l, we can determine k and l so that $\alpha + k\beta + l\gamma$† shall pass through any two points, and therefore $\alpha + k\beta + l\gamma$ must coincide with the equation of the line joining the two points. We substitute in the equation

$$\alpha + k\beta + l\gamma = 0$$

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* The demonstrations in this article are taken from a memoir by M. Plucker—(*Crelle*, vol. v. p. 274.)

† It is evident that the equation $l\alpha + m\beta + n\gamma$, though it appears to contain a constant more, is not in reality more general than the equation $\alpha + k\beta + l\gamma = 0$, for, dividing by the coefficient of α, it is reduced to the same form,

$$\alpha + \frac{m}{l}\beta + \frac{n}{l}\gamma = 0.$$

the co-ordinates of the two points, and we get two conditions

$$a' + k\beta' + l\gamma' = 0, \text{ and } a'' + k\beta'' + l\gamma'' = 0,$$

which are sufficient to determine k and l (a', &c., being what a or $x\cos a + y\sin a - p$ becomes, when x' and y' are substituted for x and y).*

63. There is one case, however, in which it would at first appear impossible to affix any meaning to the equation

$$a + k\beta + l\gamma = 0.$$

Let the sides of the triangle formed by the lines α, β, γ, be a, b, c, and let us inquire what line is represented by the equation

$$a\alpha + b\beta + c\gamma = 0.$$

Since α denotes the perpendicular from any point O on a, $a\alpha$ is double the area of the triangle OBC; in like manner, $b\beta$ is double OAC, and $c\gamma$ is double OAB; therefore, $a\alpha + b\beta + c\gamma$ is always double the triangle ABC, and can never = 0, if O be any point within the triangle. Nor is the difficulty removed by supposing the point O outside the triangle. If, for instance, the point O were below the base, we must then give a negative sign to the perpendicular α, and the quantity $a\alpha + b\beta + c\gamma$ will then be = 2 (OAC + OAB − OBC), that is, still = twice the area of the triangle ABC. It appears, therefore, that no point can be found either within or without the triangle to satisfy the equation

$$a\alpha + b\beta + c\gamma = 0.$$

64. To solve this difficulty, let us return to the general equation of the right line, $Ax + By + C = 0$. We saw that the intercepts which this line cuts off on the axes are $-\frac{C}{A}$, $-\frac{C}{B}$; consequently, the smaller A and B become, the greater will be the intercepts on the axis, and, therefore, the more remote the line represented by $Ax + By + C = 0$. Let A and B be both = 0, then the intercepts become infinite, and the line is altogether situated at an infinite distance from the origin. Hence we arrive at the

* It is worth remarking that the demonstration of Art. 11 applies equally to prove that the length of the perpendicular on the line α, from the point which cuts the line joining these points in the ratio $l : m = \frac{l\alpha'' + m\alpha'}{l + m}$.

conclusion, that the paradoxical equation $C = 0$, a constant $= 0$, represents a right line situated altogether at an infinite distance from the origin.

The equation, therefore,

$$a\alpha + b\beta + c\gamma = 0;$$

or, which is the same thing,

$$a \sin A + \beta \sin B + \gamma \sin C$$

(where sin A, &c., are the sines of the angles opposite a, &c.), is only another form of the equation $C = 0$, and is, therefore, the equation of a line situate altogether at infinity.

65. We saw (Art. 38) that the equations of two parallel lines only differ in the constant part of their equations, or that a line parallel to the line $a = 0$ has an equation of the form $a + C = 0$. Now the last Article shows that this is only an additional illustration of the principle of Art. 48. For, a parallel to a may be considered as intersecting it at an infinite distance, but (Art. 48) an equation of the form $a + C = 0$ represents a line through the intersection of the lines $a = 0$, $C = 0$, or (Art. 64) through the intersection of the line a with the line at infinity.

Hence we can express the equation of a line drawn through a point $a\beta$ parallel to a line $\gamma = 0$, namely,

$$a \sin A + \beta \sin B = 0,$$

for this line evidently passes through the point $a\beta$, and it also passes through the intersection of γ with the line at infinity,

$$a \sin A + \beta \sin B + \gamma \sin C.$$

66. We shall now prove, by these methods, some of the general properties of triangles, already investigated in the last chapter.

The three bisectors of the sides of a triangle meet in a point. We know that a parallel to the base of a triangle, drawn through vertex, forms an harmonic pencil with the bisector of the base and the sides; therefore, since the equation of the parallel is (Art. 65)

$$a \sin A + \beta \sin B = 0,$$

therefore (Art. 54), the equation of the bisector is

$$a \sin A - \beta \sin B = 0$$

(or the truth of this equation may be seen directly, since the ratio

of the perpendiculars on sides from the point where bisector meets base, plainly is sin A : sin B). The equations, then, of the three bisectors being

$$\alpha \sin A - \beta \sin B = 0,\ \beta \sin B - \gamma \sin C = 0,\ \gamma \sin C = \alpha \sin A = 0$$

(Art. 51), these lines all pass through the same point.

The three perpendiculars of a triangle meet in a point. The perpendicular of a triangle divides the vertical angle into parts, which are the complements of the base angles; hence (Art. 53) its equation is

$$\alpha \cos A - \beta \cos B = 0,$$

and this must (Art. 51) pass through the same point with

$$\beta \cos B - \gamma \cos C = 0,\quad \gamma \cos C - \alpha \cos A = 0.$$

67. *If there be two triangles such that the perpendiculars from the vertices of one on the sides of the other meet in a point, then,* vice versâ, *the perpendiculars from the vertices of the second on the sides of the first will meet in a point.*

Let the sides be $\alpha, \beta, \gamma,\ \alpha', \beta', \gamma'$, and let us denote by $(\alpha\beta)$ the angle between α and β.

Then the equation of the perpendicular

$$\text{from } \alpha\beta \text{ on } \gamma' \text{ is } \alpha \cos(\beta\gamma') - \beta \cos(\alpha\gamma') = 0,$$
$$\text{from } \beta\gamma \text{ on } \alpha' \text{ is } \beta \cos(\gamma\alpha') - \gamma \cos(\beta\alpha') = 0,$$
$$\text{from } \alpha\gamma \text{ on } \beta' \text{ is } \gamma \cos(\alpha\beta') - \alpha \cos(\gamma\beta') = 0.$$

The condition that these should meet in a point is found by eliminating β between the first two, and examining whether the resulting equation coincides with the third. It is

$$\cos(\alpha\beta') \cos(\beta\gamma') \cos(\gamma\alpha') = \cos(\alpha'\beta) \cos(\beta'\gamma) \cos(\gamma'\alpha).$$

But the symmetry of this equation shows that this is also the condition that the perpendiculars from the vertices of the second triangle on the sides of the first should meet in a point.

68. We shall hereafter frequently have occasion to use the equation of a line passing through two points given by such equations as

$$(k\alpha - \beta = 0,\ l\alpha - \gamma = 0),\ \text{and}\ (k'\alpha - \beta = 0,\ l'\alpha - \gamma = 0).$$

The equation must be of the form

$$k\alpha - \beta + A(l\alpha - \gamma) = 0,\ \text{and also}\ k'\alpha - \beta + A'(l'\alpha - \gamma) = 0.$$

Comparing these we have

$$A = A' = -\frac{k - k'}{l - l'},$$

and the equation is

$$(kl' - k'l)\alpha + (l - l')\beta - (k - k')\gamma = 0.$$

This equation is that of a line through the two points, for it is satisfied either by putting

$$\beta = k\alpha, \ \gamma = l\alpha, \text{ or } \beta = k'\alpha, \ \gamma = l'\alpha.$$

69. We shall conclude this chapter by applying the principle established in Article 48 to the solution of an important class of questions, namely, when it is required to prove that a right line, whose position is not fully determined by the conditions of the question, still passes through a fixed point. We saw there, that the line

$$Ax + By + C + k.(A'x + B'y + C) = 0;$$

or, what is the same thing,

$$(A + kA')x + (B + kB')y + C + kC' = 0,$$

where k is indeterminate, always passes through a fixed point, namely, the intersection of the lines

$$Ax + By + C, \text{ and } A'x + B'y + C'.$$

Hence, conversely, if the equation of a right line contain an indeterminate quantity *in the first degree*, the right line will always pass through a fixed point. For example:

Ex. 1. *Given vertical angle of a triangle and the sum of the reciprocals of the sides, the base will always pass through a fixed point.*

Take the sides for axes, and (Art. 30) equation of base is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

and we are given the condition

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}, \text{ or } \frac{1}{b} = \frac{1}{c} - \frac{1}{a},$$

therefore, equation of base is

$$\frac{x}{a} + \frac{y}{c} - \frac{y}{a} = 1,$$

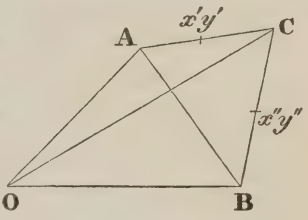
where c is constant and a indeterminate, that is,

$$\frac{1}{a}(x - y) + \frac{y}{c} - 1 = 0,$$

where $\frac{1}{a}$ is indeterminate. Hence the base must always pass through the intersection of the two lines

$$x - y = 0, \text{ and } \frac{y}{c} - 1 = 0.$$

Ex. 2. *Given three fixed lines,* OA, OB, OC, *meeting in a point, if the three vertices of a triangle move one on each of these lines, and two sides of the triangle pass through fixed points, to prove that the remaining side passes through a fixed point.*



Take for axes the fixed lines OA, OB, on which the base angles move, then the line on which the vertex moves will have an equation of the form $y = mx$, and let the fixed points be $x'y'$, $x''y''$. Now, in any position of the vertex, let its co-ordinates be $x = a$, and, consequently, $y = ma$; then the equation of AC is

$$(x' - a)\,y - (y' - ma)\,x + a\,(y' - mx') = 0.$$

Similarly, the equation of BC is

$$(x'' - a)\,y - (y'' - ma)\,x + a\,(y'' - mx'') = 0.$$

Now, the length of the intercept OA is found by making $x = 0$ in equation AC, or

$$y = -\frac{a(y' - mx')}{x' - a}.$$

Similarly, OB is found by making $y = 0$ in BC, or

$$x = \frac{a(y'' - mx'')}{y'' - ma}.$$

Hence, from these intercepts, equation of AB is

$$x\frac{y'' - ma}{y'' - mx''} - y\frac{x' - a}{y' - mx'} = a.$$

But since a is indeterminate, and only in the first degree, this line always passes through a fixed point. The particular point is found by arranging the equation in the form

$$\frac{y''}{y'' - mx''}x - \frac{x'}{y' - mx'}y - a\left(\frac{mx}{y'' - mx''} - \frac{y}{y' - mx'} + 1\right) = 0.$$

Hence the line passes through the intersection of the two lines

$$\frac{y''}{y'' - mx''}x - \frac{x'}{y' - mx'}y = 0,$$

and

$$\frac{mx}{y'' - mx''} - \frac{y}{y' - mx'} + 1 = 0.$$

Ex. 3. *If in the last example the line on which the vertex* C *moves do not pass through* O, *to determine whether in any case the base will pass through a fixed point.*

We retain the same axes and notation as before, with the only difference that the equation of the line on which C moves will be $y = mx + n$, and the co-ordinates of the vertex in any position will be a, and $ma + n$. Then the equation of AC is

$$(x' - a)\,y - (y' - ma - n)\,x + a\,(y' - mx') - nx' = 0.$$

The equation of BC is

$$(x'' - a)\,y - (y'' - ma - n)\,x + a\,(y'' - mx'') - nx'' = 0,$$

$$\text{OA} = -\frac{a\,(y' - mx') - nx'}{x' - a}; \quad \text{OB} = \frac{a\,(y'' - mx'') - nx''}{y'' - ma - n}.$$

The equation of AB is therefore

$$x \cdot \frac{y'' - ma - n}{a\,(y'' - mx'') - nx''} - y \cdot \frac{x' - a}{a\,(y' - mx') - nx'} = 1.$$

Now when this is cleared of fractions, it will in general contain a in the second degree, and, therefore, the base will in general not pass through a fixed point; *if, however, the points* $x'y'$, $x''y''$, *lie in a right line* $(y = kx)$ *passing through* O, we may substitute in the denominators $y'' = kx''$, and $y' = kx'$, and the equation becomes

$$x \cdot \frac{y'' - ma - n}{x''} - y \cdot \frac{x' - a}{x'} = a\,(k - m) - n,$$

which only contains a *in the first degree*, and, therefore, denotes a right line passing through a fixed point.

70. Ex. 4. *If a line be such that the sum of the perpendiculars let fall on it from a number of fixed points, each multiplied by a constant, may* = 0, *it will pass through a fixed point.*

Let the equation of the line be

$$x \cos \alpha + y \sin \alpha - p = 0,$$

then the perpendicular on it from $x'y'$ is

$$x' \cos \alpha + y' \sin \alpha - p,$$

and the conditions of the problem give us

$$m'(x' \cos \alpha + y' \sin \alpha - p) + m''(x'' \cos \alpha + y'' \sin \alpha - p) + m'''(x''' \cos \alpha + y''' \sin \alpha - p) + \&c. = 0,$$

or $$\Sigma(mx') \cos \alpha + \Sigma(my') \sin \alpha - p\Sigma(m) = 0,$$

using the abbreviations $\Sigma(mx')$ for the sum* of the mx, that is,

$$m'x' + m''x'' + m'''x''' + \&c.$$

In like manner $\Sigma(my')$ for

$$m'y' + m''y'' + m'''y''' + \&c.,$$

and $\Sigma(m)$ for the sum of the m's or

$$m' + m'' + m''' + \&c.$$

Hence we get

$$p\Sigma(m) = \Sigma(mx') \cos \alpha + \Sigma(my') \sin \alpha.$$

Substituting the value of p in the original equation, we get for the equation of the moveable line

$$x\Sigma(m) \cos \alpha + y\Sigma(m) \sin \alpha - \Sigma(mx') \cos \alpha - \Sigma(my') \sin \alpha = 0,$$

or $$x\Sigma(m) - \Sigma(mx') + \{y\Sigma(m) - \Sigma(my')\} \tan \alpha = 0.$$

Now as this equation involves the indeterminate $\tan \alpha$ in the first degree, it passes through the fixed point determined by the equations

$$x\Sigma(m) - \Sigma(mx') = 0, \text{ and } y\Sigma(m) - \Sigma(my') = 0,$$

or, writing at full length,

$$x = \frac{m'x' + m''x'' + m'''x''' + \&c.}{m' + m'' + m''' + \&c.} \qquad y = \frac{m'y' + m''y'' + m'''y''' + \&c.}{m' + m'' + m''' + \&c.}$$

This point has sometimes been called the *centre of mean position* of the given points.

71. If the equation of any line involve the co-ordinates of a certain point in the first degree, thus,

$$(Ax' + By' + C)x + (A'x' + B'y' + C')y + (A''x' + B''y' + C'') = 0.$$

Then if the point $x'y'$ move along a right line, the line whose equation has just been written will always pass through a fixed point. For, suppose the point always to lie on the line

$$Lx' + My' + N = 0,$$

* By sum we mean the *algebraic* sum, for any of the quantities $m'm''$, &c., may be negative.

then, if by the help of this relation, we eliminate x' from the given equation, the indeterminate y' will remain in it of the first degree, therefore the line will pass through a fixed point.

Or, again, *if the coefficients in the equation* $Ax + By + C = 0$, *be connected by the relation* $aA + bB + cC = 0$ (*where* a, b, c, *are constant and* A, B, C, *may vary*) *the line represented by this equation will always pass through a fixed point.*

For by the help of the given relation we can eliminate C and write the equation

$$(cx - a)\,A + (cy - b)\,B = 0,$$

a right line passing through the point $\left(x = \frac{a}{c},\quad y = \frac{b}{c}\right)$.

CHAPTER V.

EQUATIONS ABOVE THE FIRST DEGREE REPRESENTING RIGHT LINES.

72. Before proceeding to speak of the *curves* represented by equations above the first degree, we shall examine some cases where these equations represent *right lines.*

If we take the equations of any two lines and multiply them together, the product will represent the two right lines, since it is rendered = 0 by any values of the co-ordinates which render either of its factors = 0. Conversely, if any equation of the second degree can be resolved into two factors of the first degree, it represents two right lines.

The simplest example of this is the equation $xy = 0$, which, being satisfied by either of the suppositions $x = 0$ or $y = 0$, is evidently the equation of the two axes of co-ordinates. Again, $x^2 - y^2 = 0$ is divisible into the factors $x - y = 0$, $x + y = 0$, and is, therefore, the equation of the two bisectors of the angles between the axes (Art. 56). More generally, let us take the equation

$$x^2 - pxy + qy^2 = 0.$$

Divide this by y^2, and it is reduced to the form

$$\left(\frac{x}{y}\right)^2 - p\frac{x}{y} + q = 0,$$

This quadratic can be resolved into the factors

$$\left(\frac{x}{y} - a\right)\left(\frac{x}{y} - b\right) = 0,$$

a and b being its roots. Hence the original equation, being the product of the two factors $(x - ay)\ (x - by)$ represents the two lines signified by the equations $x - ay = 0$, $x - by = 0$, both of which pass through the origin.

73. Let us consider more minutely the three cases where the roots of this equation are real and unequal, real and equal, or both imaginary.

The first case presents no difficulty: a and b are the tangents of the angles which the lines make with the axis of y (the axes being supposed rectangular), p is therefore the sum of those tangents, and q their product.

In the second case, when $a = b$, it was once usual among geometers to say that the equation represented but one right line $(x - ay = 0)$. We shall find, however, many advantages in making the language of geometry correspond exactly to the language of algebra, and as we do not say that the equation above has *only one* root, but that it has *two equal* roots, so we shall not say that it represents *only one* line, but that it represents *two coincident* right lines.

Thirdly, let the roots be both imaginary.

It was usual to say, that in this case the equation does not represent any right lines, for no real co-ordinates can be found to satisfy it, except the co-ordinates of the origin $x = 0$, $y = 0$; hence it was said that in this case the equation was the equation of the origin. Now this language appears to us very objectionable, for we saw (Arts. 17, 18) that *two* equations are required to determine any point, hence we are unwilling to acknowledge *any single* equation as the equation of a point. Moreover, we have been hitherto accustomed to find that two *different* equations always had *different* geometrical significations, but here we should have innumerable equations, all purporting to be the equation of the same point ; for it is obviously immaterial what the values of p and q are, provided only that they give imaginary values for the roots, or that p^2 be less than $4q$. We think it, therefore,

much preferable to make our language correspond exactly to the language of algebra; and as we do not say that the equation above has *no* roots when p^2 is less than $4q$, but that it has two *imaginary* roots, so we shall not say that, in this case, it represents *no* right lines, but that it represents two *imaginary* right lines. In short the equation $x^2 - pxy + qy^2 = 0$ being *always* reducible to the form $(x - ay)\ (x - by) = 0$, we shall always say that it represents two right lines drawn through the origin; but when a and b are real, we shall say that these lines are real; when a and b are equal, we shall say that the lines coincide; and when a and b are imaginary, that the lines are imaginary. It may seem to the student a matter of indifference which mode of speaking we adopt; we shall find, however, as we proceed, that we should lose sight of many important analogies by refusing to adopt the language here recommended.

Similar remarks apply to the equation

$$Ax^2 + Bxy + Cy^2 = 0,$$

which can be reduced to the form $x^2 - pxy + qy^2 = 0$, by dividing by the coefficient of x^2. This equation will always represent two right lines through the origin; these lines will be real if $B^2 - 4AC$ be positive, as at once appears from solving the equation; they will coincide if $B^2 - 4AC = 0$; and they will be imaginary if $B^2 - 4AC$ be negative.

74. *To find the angle contained by the lines represented by the equation* $x^2 - pxy + qy^2 = 0$.

Let this equation be equivalent to $(x - ay)\ (x - by) = 0$, then the tangent of the angle between the lines is (Art. 38) $\frac{a - b}{1 + ab}$, but the product of the roots of the given equation $= q$, and the difference $= \sqrt{(p^2 - 4q)}$. Hence

$$\tan \phi = \frac{\sqrt{(p^2 - 4q)}}{1 + q}.$$

If the equation had been given in the form

$$Ax^2 + Bxy + Cy^2 = 0,$$

it will be found that

$$\tan \phi = \frac{\sqrt{(B^2 - 4AC)}}{A + C}.$$

Hence the lines will coincide, or $\tan \phi$ will $= 0$, if $p^2 = 4q$ in the first case, or $B^2 = 4AC$ in the second.

The lines will cut at right angles, or $\tan \phi$ will become infinite, if $q = -1$ in the first case, or if $A + C = 0$ in the second.

75. It will be a useful exercise on the preceding Articles, to find the equation which will represent the lines bisecting the angles between the lines represented by the equation

$$Ax^2 + Bxy + Cy^2 = 0.$$

Let these lines be $x - ay = 0$, $x - by = 0$; let the equation of the bisector be $x - \mu y = 0$, and we seek to determine μ. Now it is plain (Art. 27) that μ is the tangent of the angle made by this bisector with the axis of y, and that this angle is half the sum of the angles made with this axis by the lines themselves. Equating, therefore, tangent of twice this angle to tangent of sum, we get

$$\frac{2\mu}{1 - \mu^2} = \frac{a + b}{1 - ab};$$

but, from the theory of equations,

$$a + b = -\frac{B}{C}, \qquad ab = \frac{A}{C};$$

therefore

$$\frac{2\mu}{1 - \mu^2} = -\frac{B}{C - A},$$

or

$$\mu^2 + 2\frac{A - C}{B}\mu - 1 = 0.$$

This gives us a quadratic to determine μ, one of whose roots will be the tangent of the angle made with the axis of y by the *internal* bisector of the angle between the lines, and the other the tangent of the angle made by the *external* bisector. We can find the combined equation of both lines by substituting in the last quadratic for μ its value $= \frac{x}{y}$, and we get

$$x^2 + 2\frac{A - C}{B}xy - y^2 = 0,*$$

* It is remarkable that the roots of this last equation will always be real, even if the roots of the equation $Ax^2 + Bxy + Cy^2 = 0$ be imaginary, which leads to the curious result, that a pair of imaginary lines may have a pair of real lines bisecting the angle between them. It is the existence of such relations between real and imaginary lines, which makes the consideration of the latter profitable.

and the form of this equation shows that the bisectors cut each other at right angles (Art. 74).

The student may also obtain this equation by forming (Art. 56) the equations of the internal and external bisectors of the angle between the lines $x - ay = 0$, $x - by = 0$, multiplying these together, and then substituting for $a + b$, and ab their values in terms of A, B, C.

76. What has been here said of equations of the second degree can easily be extended to equations of any degree. If we take any number of equations and multiply them together, the compound equation will represent the aggregate of all the lines represented by its factors; for it will be satisfied by the values of the co-ordinates which make any of its factors = 0: and, conversely, *if an equation of any degree can be resolved into others of lower degrees, it will represent the aggregate of all the loci represented by its different factors.* If an equation of the n^{th} degree, for example, can be resolved into n factors of the first degree, it will represent n right lines. An instance of this occurs in the homogeneous equation of the n^{th} degree,

$$x^n - px^{n-1}y + qx^{n-2}y^2 - \&c. \ldots + ty^n = 0.$$

Divide by y^n, and we get

$$\left(\frac{x}{y}\right)^n - p\left(\frac{x}{y}\right)^{n-1} + q\left(\frac{x}{y}\right)^{n-2} - \&c. = 0.$$

Hence the equation is equivalent to

$$(x - ay)\ (x - by)\ (x - cy)\ (\&c.) = 0,$$

where a, b, c are the roots of the last equation. The homogeneous equation, therefore, represents n right lines passing through the origin. The same remarks concerning equal or imaginary roots apply as in Art. 73.

In precisely a similar manner, it may be seen that the equation

$$(x-a)^n - p(x-a)^{n-1}(y-b) + q(x-a)^{n-2}(y-b)^2 - \&c. + t(y-b)^n = 0$$

represents n real or imaginary right lines passing through the point $(x = a,\ y = b)$.

77. We have seen (Art. 73) that equations of the second degree will, *in some cases*, represent right lines; they will not,

however, *always* represent right lines, because an equation of the second degree between two variables cannot *in general* be resolved into two factors of the first degree. In order that this should be possible, it is necessary that a certain relation should be satisfied by the co-efficients of the equation. For example, suppose we were given the general equation of the second degree,

$$1 + Ax + By + Cx^2 + Dxy + Ey^2 = 0,$$

and that we examine whether this can be identical with the product of the equations of any two right lines,

$$(1 + ax + by)\,(1 + a'x + b'y) = 0$$

(we have made the absolute term in each case = 1, in order to facilitate the comparison of the equations); multiplying out this last product, and equating the co-efficients of each term to the corresponding co-efficients of the first equation, we obtain five conditions, which must be satisfied, in order that the equation

$$1 + Ax + By + Cx^2 + Dxy + Ey^2 = 0,$$

should be resolvable into the factors

$$(1 + ax + by)\,(1 + a'x + b'y).$$

Now a, a', b, b' being arbitrary, we can find their values so as to satisfy four of these conditions, and if we substitute these values in the fifth equation, we shall obtain a relation which must be satisfied by the co-efficients of the equation of the second degree, in order that it should represent two right lines.*

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* We have indicated in the text the mode of proceeding which is most easily extended to equations of higher degrees. In the case of the second degree, however, it is more simple to solve the equation directly for x or y. Thus, solving for x the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

we have

$$x = -\frac{By + D \pm \sqrt{\{(B^2 - 4AC)\,y^2 + 2\,(BD - 2AE)\,y + D^2 - 4AF\}}}{2A}.$$

In order that this may be capable of being reduced to the form $x = my + n$, it is necessary that the quantity under the radical should be a perfect square, and the equation will denote two right lines according to the different signs we give the radical. But the condition that the quantity under the radical should be a perfect square, is

$$(B^2 - 4AC)\,(D^2 - 4AF) = (BD - 2AE)^2,$$

or expanding and dividing by 4A,

$$AE^2 + CD^2 + FB^2 - BDE - 4ACF = 0.$$

78. The general equation of the n^{th} degree will not represent n right lines, unless a still greater number of conditions be fulfilled.

If it were required to ascertain the number of these conditions, we should proceed exactly as in the last article. We should multiply together the equations of n right lines,

$$(1 + ax + by)\ (1 + a'x + b'y)\ (1 + a''x + b''y)\ (\&c.) = 0,$$

and equate the co-efficients of the terms of the product to the corresponding co-efficients in the general equation of the n^{th} degree. In order to see how many equations this would give us, we must examine how many terms there are in the general equation of the n^{th} degree. Put it into the form

$$\begin{gathered} 1 \\ + Ax + By \\ + Cx^2 + Dxy + Ey^2 \\ + Fx^3 + Gx^2y + Hxy^2 + Ky^3 \\ + \&c., \end{gathered}$$

and it is plain that the number of terms in the equation is the sum of the arithmetic series

$$1 + 2 + 3 + 4 \ldots + (n + 1) = \frac{(n + 1)\ (n + 2)}{1\ .\ 2}.$$

Now the number of conditions arising from the comparison of the two equations is one less than this, since the first term, 1, is the same in both ; therefore, the number of conditions

$$= \frac{(n + 1)\ (n + 2)}{1\ .\ 2} - 1 = \frac{n\ (n + 3)}{1\ .\ 2}.$$

Now there are two arbitrary quantities in the equation of each of the n right lines, which will enable us to satisfy $2n$ of these conditions. Hence the number of relations remaining to be satisfied by the co-efficients of the equation of the n^{th} degree is

$$\frac{n\ (n + 3)}{1\ .\ 2} - 2n = \frac{n\ (n - 1)}{1\ .\ 2}.$$

This is the condition that the general equation of the second degree should represent two right lines. It might also, in like manner, have been expressed

$$(B^2 - 4AC)\ (E^2 - 4CF) - (BE - 2CD)^2 = 0,$$

or

$$(D^2 - 4AF)\ (E^2 - 4CF) - (DE - 2BF)^2 = 0.$$

We shall afterwards mention other modes of obtaining it.

79. If it were required to find how many conditions should be fulfilled, that the equation of the n^{th} degree should represent n right lines, *all passing through a given point.* Beside the conditions found in the last Article, we shall have an additional one for each line which is required to pass through a fixed point; the number, therefore, is

$$\frac{n \,.\, n-1}{1 \,.\, 2} + n = \frac{n(n+1)}{1 \,.\, 2}.$$

CHAPTER VI.

THE CIRCLE.

80. Before we proceed to the general investigation of the curves represented by the complete equation of the second degree, it seems desirable that we should examine the equation of the *circle*, which ranks next to that of the right line in simplicity.

In order to find its equation we have only to express analytically its fundamental property, viz., that the distance of any point on the curve from the centre is constant.

Let the axes be rectangular, and the co-ordinates of the centre $x = a$, $y = b$, then (Art. 9) the square of the distance of any point from the centre is

$$(x - a)^2 + (y - b)^2,$$

and, putting this equal to the square of the radius, we get for the equation of the curve

$$(x - a)^2 + (y - b)^2 = r^2.$$

If the axes were oblique we should obtain in like manner for the equation of the circle (Art. 10),

$$(x - a)^2 + (y - b)^2 + 2(x - a)\,(y - b)\cos\omega = r^2.$$

We shall, however, rarely employ oblique co-ordinates in questions relating to circles.

81. By comparing these forms with the general equation of the second degree,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

we can ascertain the condition that this latter equation should represent a circle.

If the axes be rectangular it is evident that B must = 0 and A = C, in order that when we divide by A the equation may be capable of being put into the form

$$(x-a)^2+(y-b)^2=r^2, \text{ or } x^2+y^2-2ax-2by+a^2+b^2-r^2=0.$$

If the axes be oblique we must compare the general equation with the equation

$$(x-a)^2+(y-b)^2+2(x-a)\,(y-b)\cos\omega=r^2,$$

and we find that in this case the general equation will represent a circle, if A = C, and $\frac{B}{A}=2\cos\omega$.

If the general equation of the second degree, referred to rectangular axes, fulfil the conditions B = 0, A = C, *we can find the radius of the circle represented by it, and also the co-ordinates of its centre,* thus fully determining the circle, both in magnitude and position; for, comparing the equations,

$$x^2+y^2+\frac{D}{A}x+\frac{E}{A}y+\frac{F}{A}=0,$$

and

$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0,$$

we get

$$\frac{D}{A}=-2a,\quad \frac{E}{A}=-2b,\quad \frac{F}{A}=a^2+b^2-r^2;$$

and, therefore,

$$a=-\frac{D}{2A},\quad b=-\frac{E}{2A},\quad r^2=\frac{D^2+E^2-4AF}{4A^2};$$

and the general equation is equivalent to

$$\left(x+\frac{D}{2A}\right)^2+\left(y+\frac{E}{2A}\right)^2=\frac{D^2+E^2-4AF}{4A^2}.$$

Since the coefficient F is not used in determining the co-ordinates of the centre, a, b, we learn that *two circles will be concentric if their equations only differ in their constant term.*

82. We shall here give some of the particular forms of the equation of the circle which occur most frequently in practice.

Firstly. Let the centre be the origin. The co-ordinates a and b in Article 80 become both = 0, and the equation is $x^2+y^2=r^2$.

This is the simplest form of the equation of the circle. It expresses (Art. 9) that the distance of any point on the curve from the origin is constant.

Secondly. Let the centre lie on either of the axes. If the centre lie on the axis of y we get a, the x of the centre, $= 0$. Hence the equation of the curve is

$$x^2 + (y - b)^2 = r^2;$$

or, if the centre lie on the axis of x, the equation is

$$(x - a)^2 + y^2 = r^2.$$

By observing the values of the co-ordinates of the centre given in the last section, we see that the equation

$$x^2 + y^2 + Dx + Ey + F = 0$$

will represent a circle having its centre on the axis of y if $D = 0$, and on the axis of x if $E = 0$.

Thirdly. Let the origin lie on the circle. In this case the term F in the general equation will be $= 0$, or the equation will be of the form

$$x^2 + y^2 + Dx + Ey = 0,$$

for it is evident that this equation will be satisfied by the values $x = 0$, $y = 0$, that is, by the co-ordinates of the origin.* The same thing will be indicated by the equation

$$(x - a)^2 + (y - b)^2 = r^2, \text{ if } a^2 + b^2 = r^2,$$

that is, if the distance of the centre from the origin be equal to the radius.

Fourthly. Let the axes be a diameter and a perpendicular to it through either extremity. In this case the two last suppositions are combined, for we have both the origin lying on the circle, and also the centre on one of the axes. Hence in the general equation we must both have $F = 0$, and also either D or $E = 0$, and the equation will be $x^2 + y^2 + Dx = 0$, if the diameter be taken for the axis of x, or $x^2 + y^2 + Ey = 0$, if the diameter be taken for the axis of y.

In this case the co-ordinates of the centre are either

$$x = r \text{ and } y = 0, \text{ or else, } x = 0 \text{ and } y = r.$$

* The same argument will prove, in general, that an equation of any degree, wanting the absolute term, will represent a curve *passing through the origin.*

Using these values for a and b in the equation

$$(x-a)^2+(y-b)^2=r^2,$$

we obtain, for the equation of the circle,

$$x^2+y^2-2rx=0, \text{ or else, } x^2+y^2-2ry=0.$$

This form of the equation of the circle is only second in simplicity and usefulness to that first given in this Article.

Fifthly. Let the radius of the circle = 0. We have seen (Art. 81) that the square of the radius of the circle represented by the equation $Ax^2+Ay^2+Dx+Ey+F=0$ is

$$\frac{D^2+E^2-4AF}{4A^2};$$

hence, if $D^2+E^2=4AF$, the radius will = 0, and the equation may be put into the form

$$(x-a)^2+(y-b)^2=0,$$

a and b being the co-ordinates of the centre.

It is plain, that this equation can be satisfied by the co-ordinates of no point save those of the point $(x=a, y=b)$; hence it has been common to say, that the equation

$$(x-a)^2+(y-b)^2=0$$

is the equation of this point. We think it more accurate to say, that it is the equation *of an infinitely small circle*, having for centre the point (ab), though we may occasionally use the shorter expression. We have seen (Art. 76) that it may also be considered as the equation of *two imaginary right lines* passing through the point (ab), since it can be resolved into the factors

$$(x-a)+(y-b)\sqrt{(-1)}=0, \text{ and } (x-a)-(y-b)\sqrt{(-1)}=0.$$

These remarks, of course, apply, in like manner, to the equation

$$x^2+y^2=0,$$

which is a particular case of the above.

Lastly. In the general equation,

$$Ax^2+Ay^2+Dx+Ey+F=0,$$

let D^2+E^2 be less than $4AF$; the radius of the circle becomes imaginary, and the equation, being equivalent to one of the form

$$(x-a)^2+(y-b)^2+r^2=0,$$

cannot be satisfied by any real values of the co-ordinates x and y.

83. We next proceed to show how to form the equations of some of the remarkable lines connected with the circle, and examine in the first place the equation of the *tangent.*

We shall in general define the tangent to any curve as the line joining two indefinitely near points on the curve, and its equation will be found by first forming the equation of the line joining any two points $(x'y', x''y'')$ on the curve, and then making $x' = x''$ and $y' = y''$ in that equation.

To apply this to the circle: first, let the centre be the origin, and, therefore, the equation of the circle $x^2 + y^2 = r^2$.

The equation of the line joining any two points $(x'y')$ and $(x''y'')$ is (Art. 33),

$$\frac{y - y'}{x - x'} = \frac{y' - y''}{x' - x''};$$

now if we were to make in this equation $y' = y''$ and $x' = x''$, the right hand member would assume the indeterminate form of $\frac{0}{0}$. The cause of this is, that we have not yet introduced the condition, that the two points $(x'y', x''y'')$ are *on the circle.* By the help of this condition we shall be able to put the equation in a form which will not become indeterminate when the two points are made to coincide. For, since

$$r^2 = x'^2 + y'^2 = x''^2 + y''^2 \text{ we have } x'^2 - x''^2 = y''^2 - y'^2,$$

and, therefore,

$$\frac{y' - y''}{x' - x''} = -\frac{x' + x''}{y' + y''}.$$

Hence the equation of the chord becomes

$$\frac{y - y'}{x - x'} = -\frac{x' + x''}{y' + y''}.$$

And if we *now* make $x' = x''$ and $y' = y''$, we find for the equation of the tangent,

$$\frac{y - y'}{x - x'} = -\frac{x'}{y'},$$

or, reducing, and remembering that $x'^2 + y'^2 = r^2$, we get finally

$$xx' + yy' = r^2.$$

We might have obtained the equation of the tangent $(xx' + yy' = r^2)$ in another way, by forming the equation of a line through the point $x'y'$, perpendicular to the radius, whose equation is easily

seen to be $y'x = x'y$. We have preferred, however, the method actually adopted, both because it is the same as that which we shall employ in the case of other curves, and also because we wish the learner to perceive that all the properties of the circle can be deduced from its equation without a previous acquaintance with the geometrical theory of the curve; as in the present instance, where the equation just found might be used to *prove* that the tangent to a circle is perpendicular to the radius.

84. The method we have used is equally applicable if we were given the equation of the circle in its most general form referred to any axes

$$Ax^2 + Bxy + Ay^2 + Dx + Ey + F = 0,$$

where $B = 2A\cos\omega$. We form, as before, the equation of the line joining two points, and then by the help of the conditions that $x'y'$, $x''y''$ are points *on the circle*, we can get an expression for $\frac{y' - y''}{x' - x''}$, which will not become indeterminate when the two points coincide. We have the two conditions

$$Ax'^2 + Bx'y' + Ay'^2 + Dx' + Ey' + F = 0,$$
$$Ax''^2 + Bx''y'' + Ay''^2 + Dx'' + Ey'' + F = 0.$$

Subtracting them

$$A(x'^2 - x''^2) + B(x'y' - x''y'') + A(y'^2 - y''^2) + D(x' - x'') + E(y' - y'') = 0.$$

Now $$x'y' - x''y'' = x'(y' - y'') + y''(x' - x'').$$

Hence, dividing by $x' - x''$, and solving for $\frac{y' - y''}{x' - x''}$, we find

$$\frac{y' - y''}{x' - x''} = -\frac{A(x' + x'') + By'' + D}{A(y' + y'') + Bx' + E}.$$

The equation of the chord is, therefore,

$$\frac{y - y'}{x - x'} = -\frac{A(x' + x'') + By'' + D}{A(y' + y'') + Bx' + E}.$$

If the points $x'y'$, $x''y''$ coincide we have the equation of the *tangent*

$$\frac{y - y'}{x - x'} = -\frac{2Ax' + By' + D}{2Ay' + Bx' + E},$$

or, reducing, and remembering that $x'y'$ satisfies the equation of the curve,

$$(2Ax' + By' + D)x + (2Ay' + Bx' + E)y + Dx' + Ey' + 2F = 0.$$

The equation of the tangent when the axes are rectangular is found by making B = 0, in the preceding equation. If the equation had been given in the form

$$(x - a)^2 + (y - b)^2 = r^2,$$

the equation of the tangent is

$$(x - a)(x' - a) + (y - b)(y' - b) = r^2,$$

a form easily remembered from its similarity to the equation of the circle itself. This might also have been obtained by transformation of co-ordinates from the equation given in the last article.

85. *To find the co-ordinates of the points in which a given right line meets a given circle.*

Let the equation of the circle be $x^2 + y^2 = r^2$, and that of the right line $x \cos \alpha + y \sin \alpha = p$. These two equations are sufficient (Art. 18) to determine the x and y of the intersection. For example, finding the values of y from both, and equating them to each other, we get for determining x, the equation

$$\frac{p - x \cos \alpha}{\sin \alpha} = \sqrt{(r^2 - x^2)},$$

or, reducing $x^2 - 2px \cos \alpha + p^2 - r^2 \sin^2 \alpha = 0$;

hence, $x = p \cos \alpha \pm \sin \alpha \sqrt{(r^2 - p^2)}$,

and, in like manner,

$$y = p \sin \alpha \mp \cos \alpha \sqrt{(r^2 - p^2)}.$$

(The reader may satisfy himself, by substituting these values in the given equations, that the – in the value of y corresponds to the + in the value of x, and *vice versâ*).

Since we obtained a quadratic to determine x, and since every quadratic has two roots, we must, in order to make our language conform to the language of algebra, assert that *every* line meets a circle in *two* points.

86. Let us, however, examine separately the three cases of this solution :

First. If p, which is the distance of the line from the centre, be less than the radius, we get two *real* values for x and y, and the line meets the circle in two real points.

Secondly. Let $p = r$, or the distance of the line from the centre = the radius. In this case it is evident geometrically that the line is a tangent to the circle, and our analysis points to the same conclusion, since the two values of x in this case become *equal*, and so likewise will the two values of y. Consequently, the points answering to these two values, which are in general different, will in this case coincide. We shall, therefore, not say that the tangent meets the circle in only one point, but rather that it meets it in two coincident points; just as we do not say that the equation for this case

$$x^2 - 2rx\cos\alpha + r^2\cos^2\alpha = 0,$$

has only one root, but rather that it has two equal roots. This accords with the general definition of a tangent which has been already given (Art. 83).

Thirdly. Let p be greater than r. In this case it is usual to say, that the line does not meet the circle at all. Analysis, however, though it fails to furnish us with real values for x and y, yet supplies us with *imaginary* values. We shall, therefore, find it more consistent to say that in this case the line meets the circle in two imaginary points. By an imaginary point we mean nothing more than a point, one or both of whose co-ordinates are imaginary. It is a purely analytical conception. We do not attempt to represent it geometrically. But the neglect of those imaginary points would lead to as great a want of generality in our reasonings, and to as much inconvenience in our language, as if, only paying attention to the real roots of equations, we were to deny that every equation has as many roots as it has dimensions, or to assert that the equation

$$x^2 - 2px\cos\alpha = r^2\sin^2\alpha - p^2$$

has no root at all when p is greater than r.

87. If we were to form the equation of the line joining the points whose co-ordinates are

$$\begin{aligned} x &= p\cos\alpha + \sin\alpha\sqrt{(r^2 - p^2)} \\ y &= p\sin\alpha - \cos\alpha\sqrt{(r^2 - p^2)} \end{aligned}, \text{ and } \begin{aligned} x &= p\cos\alpha - \sin\alpha\sqrt{(r^2 - p^2)} \\ y &= p\sin\alpha + \cos\alpha\sqrt{(r^2 - p^2)} \end{aligned},$$

we should of course fall back on the given equation

$$x\cos\alpha + y\sin\alpha - p = 0.$$

This will be true whether the points be real or imaginary. Hence the line joining two imaginary points may be real, and we saw (Art. 73) that two imaginary lines might intersect in a real point. This circumstance increases the necessity of attending to imaginary points, for we shall presently see many cases in which the real line, joining two imaginary points, enjoys all the geometrical properties of the corresponding line in the case where the points are real.

The student will find no difficulty in determining the points where a line given by the general equation $ax + by + c = 0$ meets the circle

$$Ax^2 + Ay^2 + Dx + Ey + F = 0.$$

We only think it necessary to notice the particular case where the given line is one of the axes of co-ordinates. The points, for instance, where the axis of x meets the circle, are found by making $y = 0$ in the general equation, and are, therefore, given by the quadratic

$$Ax^2 + Dx + F = 0.$$

The intercepts on the axis of y are given by the quadratic

$$Ay^2 + Ey + F = 0.$$

Hence the axis of x will be a tangent, if

$$D^2 = 4AF,$$

and the axis of y, if $E^2 = 4AF$.

In general, the rectangle under the intercepts made on the axis of $x = \frac{F}{A}$; but this is the same value as we find for the rectangle under the intercepts on the axis of y (Euclid, III. 36).

We might, in the same manner, find the condition, that the line $Ax + By + C = 0$ should touch the circle

$$(x - a)^2 + (y - b)^2 = r^2;$$

or we can obtain it at once, by expressing the condition that the perpendicular on the line from the point ab is $= r$, that is (Art. 31)

$$\frac{Aa + Bb + C}{\sqrt{(A^2 + B^2)}} = r.$$

88. *To find the points of contact of tangents drawn to a circle from a given point.* Let the given point be $x'y'$, and let the co-

ordinates of the point of contact which we are seeking be x'', y''. Then (Art. 83) the equation of the tangent will be

$$xx'' + yy'' = r^2;$$

but by hypothesis this line passes through the point $x'y'$, hence we get the condition

$$x'x'' + y'y'' = r^2;$$

and since the point $x''y''$ is on the circle, we have also the condition

$$x''^2 + y''^2 = r^2.$$

These two conditions are sufficient to determine the co-ordinates x'', y''. Solving the equations, we get

$$x'' = \frac{r^2x' \pm ry'\sqrt{(x'^2 + y'^2 - r^2)}}{x'^2 + y'^2},$$

and

$$y'' = \frac{r^2y' \mp rx'\sqrt{(x'^2 + y'^2 - r^2)}}{x'^2 + y'^2}.$$

Hence, from every point may be drawn *two* tangents to a circle. These tangents will be real when $x'^2 + y'^2$ is $> r^2$, or the point outside the circle; they will be imaginary when $x'^2 + y'^2$ is $< r^2$, or the point inside the circle; and they will coincide when $x'^2 + y'^2 = r^2$, or the point on the circle.

89. *To find the equation of the line joining the points of contact of tangents from any point.* That is, to form the equation of the line joining the two points whose co-ordinates were found in the last article. It will not, however, be necessary to set about this in the usual manner, if we attend to the remark at the end of Art. 33. We saw in the last article that the co-ordinates of each point of contact were connected with those of the given point by the relation

$$x'x'' + y'y'' = r^2.$$

The equation, therefore, of the line joining the points of contact, must be

$$xx' + yy' = r^2,$$

for this is the equation of a right line, and is satisfied for each point of contact. The co-ordinates found in the last Article are evidently precisely the same as those which we should obtain if we sought the intersection of the circle $x'^2 + y'^2 = r^2$ with the right line $xx' + yy' = r^2$. This latter equation is exactly similar in form to the equation of the tangent, only that in the present case the point $x'y'$ need not be supposed on the curve.

It is plain that if the co-ordinates x' and y' be real, the equation $xx' + yy' = r^2$ will be real, whether the point be within or without the circle. The line it represents is familiar to the learner as the *polar* of the point $x'y'$; and we see that, when the point $x'y'$ is outside the circle, the polar is the line joining the points of contact of real tangents; and that, when the point is inside the circle, the polar must be considered as the line joining the points of contact of the imaginary tangents from the point. When the point $x'y'$ is on the circle, it is evident from the equation that its polar is the tangent.

The polar is evidently perpendicular to the line $(x'y - y'x = 0)$ joining the centre to the point $x'y'$, and its distance from the centre $= \dfrac{r^2}{\sqrt{(x'^2 + y'^2)}}$.

90. We can by the same process find the equation of the polar of any point $x'y'$ with regard to a circle expressed by its most general equation

$$Ax^2 + Bxy + Ay^2 + Dx + Ey + F = 0.$$

The equation of the tangent at any point $x''y''$, may be written (Art. 84)

$$(2Ax + By + D)\,x'' + (2Ay + Bx + E)\,y'' + Dx + Ey + 2F = 0;$$

and since this line by hypothesis passes through the point $x'y'$, the co-ordinates of each point of contact must fulfil the relation

$$(2Ax' + By' + D)\,x'' + (2Ay' + Bx' + E)\,y'' + Dx' + Ey' + 2F = 0.$$

The equation then of the line joining the points of contact must be

$$(2Ax' + By' + D)\,x + (2Ay' + Bx' + E)\,y + Dx' + Ey' + 2F = 0,$$

an equation precisely similar in form to the equation of the tangent. Hence the polar of *the origin* with regard to this circle is

$$Dx + Ey + 2F = 0.$$

We can find the co-ordinates of the *pole*, with regard to the circle $(x^2 + y^2 = r^2)$, of any line $(Ax + By + C = 0)$, by comparing this last equation with the equation $xx' + yy' = r^2$, where x', y' are the co-ordinates of the pole. Hence we shall find

$$x' = -\frac{Ar^2}{C}, \quad \text{and } y' = -\frac{Br^2}{C}.$$

91. *To find the ratio in which the line joining two given points, $x'y'$, $x''y''$, is cut by a given circle.*

We proceed precisely as in Article 43. The co-ordinates of any point on the line must (Art. 11) be of the form

$$\frac{lx'' + mx'}{m+n}, \quad \frac{ly'' + my'}{m+n}.$$

Substituting these values in the equation of the circle

$$x^2 + y^2 - r^2 = 0,$$

and arranging, we have to determine the ratio $l : m$, the quadratic

$$l^2(x''^2 + y''^2 - r^2) + 2lm(x'x'' + y'y'' - r^2) + m^2(x'^2 + y'^2 - r^2) = 0.$$

The values of $l : m$ being determined from this equation, we have at once the co-ordinates of the points where the right line meets the circle. The symmetry of the equation makes this method sometimes more convenient than that used Art. 85.

If $x''y''$ lie on the polar of $x'y'$, we have (Art. 89) $x'x'' + y'y'' - r^2 = 0$, and the roots of the preceding equation must be of the form $l + \mu m$, $l - \mu m$; the line joining $x'y'$, $x''y''$ is therefore cut internally and externally in the same ratio, and we deduce the well-known theorem, *any line drawn through a point is cut harmonically by the point, the circle, and the polar of the point.*

92. *To find the equation of the tangents from a given point to a given circle.*

We have already (Art. 88) found the co-ordinates of the point of contact; substituting, therefore, these values in the equation $xx'' + yy'' - r^2 = 0$, we have for the equation of the tangent

$$r(xx' + yy' - x'^2 - y'^2) + (xy' - yx')\sqrt{(x'^2 + y'^2 - r^2)} = 0,$$

and for the other

$$r(xx' + yy' - x'^2 - y'^2) - (xy' - yx')\sqrt{(x'^2 + y'^2 - r^2)} = 0.$$

These two equations multiplied together give the equation of the pair of tangents in a form free from radicals. The preceding article enables us, however, to obtain this equation in a still more simple form. For the equation which determines $l : m$ will have equal roots if the line joining $x'y'$, $x''y''$ touch the given circle; if

then $x''y''$ be any point on either of the tangents through $x'y'$, its co-ordinates must satisfy the condition

$$(x'^2 + y'^2 - r^2)(x^2 + y^2 - r^2) = (xx' + yy' - r^2)^2.$$

This, therefore, is the equation of the pair of tangents through the point $x'y'$. It is not difficult to prove that this equation is identical with that obtained by the method first indicated.

93. *To find the length of the tangent drawn from any point to the circle, whose equation is*

$$(x - a)^2 + (y - b)^2 - r^2 = 0.$$

The square of the distance of any point from the centre

$$= (x - a)^2 + (y - b)^2;$$

and since this square exceeds the square of the tangent by the square of the radius, the square of tangent

$$= (x - a)^2 + (y - b)^2 - r^2.$$

Hence the square of the length of the tangent from any point is found by substituting the co-ordinates of that point for x and y in the equation of the circle

$$(x - a)^2 + (y - b)^2 - r^2 = 0.$$

Since the general equation to rectangular co-ordinates

$$Ax^2 + Ay^2 + Dx + Ey + F = 0,$$

when divided by A, is (Art. 81) equivalent to one of the form

$$(x - a)^2 + (y - b)^2 = r^2.$$

We learn that the square of the tangent to a circle whose equation is given in its most general form, is found by dividing by the co-efficient of x^2, and then substituting in the equation the co-ordinates of the given point.

The square of the tangent from the origin is found by making x and $y = 0$, and is, therefore, = the absolute term in the equation of the circle.

The same reasoning is applicable if the co-ordinates be oblique.

94. We shall conclude this chapter by showing how to find the *polar equation* of a circle.

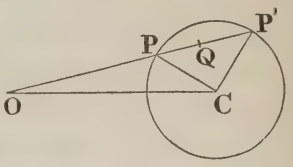
We may either obtain it, by substituting for x, $\rho \cos \theta$, and

for y, $\rho \sin \theta$ (Art. 15), in either of the equations of the circle already given,

$$Ax^2 + Ay^2 + Dx + Ey + F = 0, \text{ or } (x - a)^2 + (y - b)^2 = r^2,$$

or else we may find it independently, from the definition of the circle, as follows:

Let O be the pole, C the centre of the circle, and OC the fixed axis; let the distance $OC = d$, and let OP be any radius vector, and, therefore, $= \rho$, and the angle $POC = \theta$, then we have



$$PC^2 = OP^2 + OC^2 - 2OP \cdot OC \cos POC,$$

that is, $$r^2 = \rho^2 + d^2 - 2\rho d \cos \theta,$$

or $$\rho^2 - 2d\rho \cos \theta + d^2 - r^2 = 0.$$

This, therefore, is the polar equation of the circle.

If the fixed axis did not coincide with OC, but made with it any angle a, the equation would be, as in Art. 40,

$$\rho^2 - 2d\rho \cos (\theta - a) + d^2 - r^2 = 0.$$

If we suppose the pole on the circle, the equation will take a simpler form, for then $r = d$, and the equation will be reduced to

$$\rho = 2r \cos \theta,$$

a result which we might have also obtained at once geometrically from the property that the angle in a semicircle is right; or else by substituting for x and y their polar values in the equation (Art. 82, IV.) $$x^2 + y^2 = 2rx.$$

CHAPTER VII.

EXAMPLES ON THE CIRCLE.

95. We purpose in the present chapter to illustrate by examples the principles laid down in the preceding chapter. Having sufficiently shown, in Chapter III., how in general to apply the analytical method to the solution of problems, we do not think it necessary to enter into the subject here with equal minuteness,

and shall feel ourselves at liberty to suppress many details which can easily be supplied by the reader who has worked out the examples there given.

We commence by illustrating the applications of the conditions (Art. 81) for determining when the equation of the second degree represents a circle.

Example 1. *When will the locus of a point be a circle, if the square of its distance from the base of a triangle be in a constant ratio to the product of its distances from the sides?*

Let the equation of the base be $y = 0$, and those of the sides

$$x\cos\alpha + y\sin\alpha - p = 0, \quad x\cos\beta + y\sin\beta - p_1 = 0.$$

Then by the conditions of the question

$$(x\cos\alpha + y\sin\alpha - p)\,(x\cos\beta + y\sin\beta - p_1) = k^2y^2.$$

In order that this should represent a circle, the two conditions

$$B = 0, \text{ and } A = C,$$

give us $\sin(\alpha + \beta) = 0$; $\cos(\alpha + \beta) = k^2$.

From the first condition the sides make equal angles with the base, and the triangle is therefore isosceles. From the second we have $k^2 = 1$.

These conditions being fulfilled, the locus will be a circle, and it will be found that it will touch the two sides of the triangle at the extremities of the base. Hence we have the following property of a circle: "*If from any point on a circle perpendiculars be let fall on any two tangents and on their chord of contact, the square of the last will be equal to the rectangle under the other two.*"

Ex. 2. *When will the locus of a point be a circle if the sum of the squares of the three perpendiculars from it, on the sides of any triangle, be constant?*

Using the same notation as in the last example, the conditions of the question give us

$$(x\cos\alpha + y\sin\alpha - p)^2 + (x\cos\beta + y\sin\beta - p_1)^2 + y^2 = m^2.$$

In order that this should represent a circle we have

$$\sin 2\alpha + \sin 2\beta = 0, \quad \cos 2\alpha + \cos 2\beta = 1.$$

The first condition is satisfied if $\beta = -\alpha$, or if the triangle be isosceles. Then the second condition gives us $\cos 2\alpha = \frac{1}{2}$. Hence

(*Luby's Trig.*, p. 14) $a = 30°$. But a is the angle which the perpendicular on one of the sides makes with the base: the angle, therefore, which the side itself makes with the base will be 60°, and the triangle will be equilateral. Hence we see that in an equilateral triangle the locus of a point for which the sum of squares of perpendiculars on sides is constant will be a circle; and it will be found that the centre of this circle is the intersection of the perpendiculars of the triangle.

Ex. 3. *When will the locus of a point be a circle, if the product of perpendiculars from it on two opposite sides of a quadrilateral be in a given ratio to the product of perpendiculars from it on the other two sides.*

Let the equations of the sides of the quadrilateral be

$$x \cos a + y \sin a - p = 0,\ x \cos \beta + y \sin \beta - p_{,} = 0, \text{ \&c.},$$

then the equation of the locus will be

$$(x \cos a + y \sin a - p)\ (x \cos \gamma + y \sin \gamma - p_{,,}) = k\ (x \cos \beta + y \sin \beta - p_{,})\ (x \cos \delta + y \sin \delta - p_{,,,}).$$

The usual tests applied to this equation will give the conditions

$$\cos (a + \gamma) = k \cos (\beta + \delta),\ \sin (a + \gamma) = k \sin (\beta + \delta).$$

Squaring these equations, and adding them, we find $k = 1$; and if this condition be fulfilled, we must have

$$a + \gamma = \beta + \delta, \text{ or } a - \beta = \delta - \gamma.$$

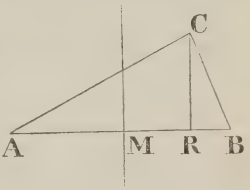
Recollecting that $a - \beta$ is the angle between the perpendilars on the lines a and β, and is, therefore, equal or supplemental to the angle between the lines themselves, it is easy to see that this condition will be fulfilled if the given quadrilateral be inscribable in a circle. Indeed from the form of the equation it may be seen, as in Art. 48, that the curve represented by the equation must pass through the angles of the given quadrilateral, and therefore cannot be a circle unless the quadrilateral be inscribable in a circle.

96. We shall next give illustrations of the method of determining the position of a circle from its equation. Beside the method (Art. 81) by finding the co-ordinates of the centre, and the radius, there is another which we may occasionally employ. For a circle is given if any three points on it are given; and if

we find, as in Art. 87, the points in which the circle meets each of the axes, we shall know four points on the circle, which will thus be completely determined.

Ex. 1. *Given base and vertical angle of a triangle, to find the locus of vertex.*

Let us take the base for axis of x, and a perpendicular through its middle point for axis of y; let the co-ordinates of the vertex be x, y, and let the base = $2c$. Then the tangent of the base angle CAB will be $\frac{CR}{AR}$, or $\frac{y}{c+x}$, and of CBR = $\frac{CR}{BR}$, or $\frac{y}{c-x}$.



Hence we can find the tangent of the sum of the base angles, and make it = − the tangent of C, the given vertical angle,

or
$$\frac{\frac{y}{c+x}+\frac{y}{c-x}}{1-\frac{y^2}{c^2-x^2}} = -\tan C,$$

and, reducing this equation, the equation of the locus will be found to be $x^2 + y^2 - 2cy \cot C - c^2 = 0$.

This (Art. 82, II.) is the equation of a circle having its centre on the axis of y, at a distance from the origin = $c \cot C$; this distance will be positive, or the centre will lie above the base, if the vertical angle be acute; it will lie on the base if the vertical angle be right, and below it if the angle be obtuse. By making $y = 0$ in this equation, we find that the circle will pass through each extremity of the base; and the radius of the circle is found = $\frac{c}{\sin C}$.

Ex. 2. *Given base and ratio of sides, find locus of vertex.*

With same axes as before, if ratio be $m : n$, we find, for equation of locus,
$$m^2\{y^2 + (c-x)^2\} = n^2\{y^2 + (c+x)^2\},$$
or
$$x^2 + y^2 - 2 \cdot \frac{m^2+n^2}{m^2-n^2} cx + c^2 = 0.$$

Hence locus is a circle, whose centre is on the axis of x, at a distance from origin = $\frac{m^2+n^2}{m^2-n^2} c$, and whose radius = $\frac{2mn}{m^2-n^2} c$.

By making $y = 0$ we find, for the co-ordinates of the points where the circle meets base,

$$x = \frac{m+n}{m-n}c, \quad \text{and } x = \frac{m-n}{m+n}c;$$

and since the co-ordinates of the extremities of base are $x = +c$, and $x = -c$, these (Art. 11) are the two points where the base is cut in the ratio of $m : n$.

Ex. 3. *Find locus of a point the square of whose distance from a given point is proportional to its distance from a given right line. Determine position and magnitude of the resulting circle.*

Ex. 4. *Given base of a triangle, and m times square of one side ± n times square of the other = a constant; find locus of vertex: find centre and radius of the resulting circle, and determine the point where it cuts base.*

Ex. 5. *A line of constant length moves between two fixed right lines, and perpendiculars to the lines are raised at its extremities, find the locus of their intersection.*

Ex. 6. *In general, given any number of points, to find locus of a point such that m′ times square of its distance from the first + m″ times square of its distance from the second + &c., = a constant: or* (*adopting the notation used in* Art. 70) *such that* $\Sigma(mr^2)$ *may be constant.*

The square of the distance of any point xy from $x'y'$ is

$$(x - x')^2 + (y - y')^2.$$

Multiply this by m', and add it to the corresponding terms found by expressing the distance of the point xy from the other points $x''y''$, &c. If we adopt the notation of Art. 70, we may write, for the equation of the locus,

$$\Sigma(m)x^2 + \Sigma(m)y^2 - 2\Sigma(mx')x - 2\Sigma(my')y + \Sigma(mx'^2) + \Sigma(my'^2) = C.$$

Hence the locus will be a circle, and, by Article 81, the co-ordinates of its centre will be

$$x = \frac{\Sigma(mx')}{\Sigma(m)}, \qquad y = \frac{\Sigma(my')}{\Sigma(m)},$$

that is to say, the centre will be the point, which, in Article 70, was called the centre of mean position of the given points.

If we investigate the value of the radius of this circle, we shall find

$$R^2\,\Sigma(m) = \Sigma(mr^2) - \Sigma(m\rho^2),$$

where $\Sigma(mr)^2 = C$ = sum of m times square of distance of each of the given points from any point on the circle, and $\Sigma(m\rho^2)$ = sum of m times square of distance of each point from centre of mean position.

Ex. 7. *Given base and vertical angle of a triangle, to find the locus of the point of intersection of the perpendiculars of the triangles.*

Take the same axes as in Example 1. We found there that the following relation existed between the co-ordinates of the vertex,

$$x^2 + y^2 - 2cy \cot C = c^2,$$

which, since the y of the vertex is the perpendicular, we may write

$$p^2 - 2pc \cot C = c^2 - x^2.$$

But we found (Art. 42, Ex. 1), that the x of the point whose locus we seek, is the same as the x of the vertex, and that its

$$y = \frac{ss'}{p}, \quad \text{or} = \frac{c^2 - x^2}{p}$$

(since $s = c + x$, and $s' = c - x$). Substitute, therefore, for p, $\frac{c^2 - x^2}{y}$; the equation will then become divisible by $c^2 - x^2$, and can be reduced to

$$x^2 + y^2 + 2cy \cot C = c^2.$$

Now this equation only differs in the sign of cot C, from that found in Example 1; hence we see that it is the locus which we should have found had we been given the same base and a vertical angle equal to the supplement of the given one; and that it is, therefore, a circle passing through the extremities of the base.

97. We come, next in order, to illustrate the use of the equations given in Art. 83, viz., of the tangent $(xx' + yy' = r^2)$, and of the chord

$$x(x' + x'') + y(y' + y'') = r^2 + x'x'' + y'y''.$$

We shall first put them into a form which is sometimes more convenient in practice. Let θ' be the angle which the radius to $x'y'$ makes with the axis of x, then $x' = r\cos\theta'$, $y' = r\sin\theta'$, and the equation of the tangent becomes

$$x\cos\theta' + y\sin\theta' = r.$$

Conversely, *if the equation of any right line can be put into the form*

$$x\cos\theta + y\sin\theta = r,$$

where r is constant, and θ indeterminate, we may infer that this right line touches the curve $x^2 + y^2 = r^2$ (see Examples 1 and 2).

If a similar substitution be made in the equation of the chord, this equation may be reduced to the form

$$x \cos \tfrac{1}{2}(\theta' + \theta'') + y \sin \tfrac{1}{2}(\theta' + \theta'') = r \cos \tfrac{1}{2}(\theta' - \theta''),$$

θ' and θ'' being the angles which radii drawn to the extremities of the chord make with the axis of x.

This equation might also have been obtained directly from the general equation of a right line (Art. 30),

$$x \cos \alpha + y \sin \alpha = p,$$

for the angle which the perpendicular on the chord makes with the axis is plainly half the sum of the angles made with the axis by radii to its extremities; and the perpendicular on the chord

$$= r \cos \tfrac{1}{2}(\theta' - \theta'').$$

Let us now apply these formulæ to examples.

Ex. 1. *If a chord of a constant length be inscribed in a circle it will always touch another circle.*

We must first express the condition, that the chord joining two points should be of a constant length. It is (Art. 9),

$$(x' - x'')^2 + (y' - y'')^2 = \text{const.}$$

or, since $x'^2 + y'^2 = r^2$, and $x''^2 + y''^2 = r^2$,

the condition may be expressed in the simpler form :

$$x'x'' + y'y'' = \text{const.}$$

This condition will take another form if we use the substitutions $r \cos \theta'$, $r \cos \theta''$, $r \sin \theta'$, $r \sin \theta''$, for x', x'', y', y''. The condition will then become

$$\cos(\theta' - \theta'') = \text{const. or } \theta' - \theta'' = \text{a const.}$$

Hence, in the equation of the chord which we have just found,

$$x \cos \tfrac{1}{2}(\theta' + \theta'') + y \sin \tfrac{1}{2}(\theta' + \theta'') = r \cos \tfrac{1}{2}(\theta' - \theta''),$$

the right hand side of the equation is constant, and the angle $\frac{1}{2}(\theta' + \theta'')$ is indeterminate; therefore, by the principle just laid down, the chord must always touch the circle whose equation is

$$x^2 + y^2 = r^2 \cos^2 \tfrac{1}{2}(\theta' - \theta''),$$

a circle concentric with the given one.

Ex. 2. *Given any number of points, if a right line be such that m' times the perpendicular on it from the first point, + m'' times the perpendicular from the second, + &c., be constant, the line will always touch a circle.*

This only differs from the question in Art. 70, in that the sum, in place of being = 0, is constant. Adopting then the notation of that Article, instead of the equation there found,

$$\{x\Sigma(m) - \Sigma(mx')\}\cos\alpha + \{y\Sigma(m) - \Sigma(my')\}\sin\alpha = 0,$$

we have only to write

$$\{x\Sigma m - \Sigma(mx')\}\cos\alpha + \{y\Sigma(m) - \Sigma(my')\}\sin\alpha = \text{const.}$$

Hence this line must always touch the curve

$$\left(x - \frac{\Sigma(mx')}{\Sigma(m)}\right)^2 + \left(y - \frac{\Sigma(my')}{\Sigma(m)}\right)^2 = \text{const.},$$

that is a circle whose centre is the centre of mean position of the given points.

Ex. 3. *Find the locus of the point where a chord of a constant length is cut in a constant ratio.*

We insert this example here, as we have just given the condition ($x'x'' + y'y'' = \text{const.}$) that a chord should be of a constant length. The co-ordinates of a point cutting the chord in a constant ratio are (Art. 11),

$$x = \frac{mx'' + nx'}{m+n}, \qquad \text{and } y = \frac{my'' + ny'}{m+n};$$

and by the help of this condition it will be found, that $x^2 + y^2$ is constant, or that the locus is a circle concentric with the given one.

Ex. 4. *If tangents be drawn at the extremities of a chord of a constant length, to find the locus of their intersection.*

The co-ordinates of the point of intersection of

$$x\cos\theta + y\sin\theta = r, \text{ and } x\cos\theta' + y\sin\theta' = r,$$

are easily found to be

$$x = r\frac{\cos\frac{1}{2}(\theta+\theta')}{\cos\frac{1}{2}(\theta-\theta')}, \qquad y = r\frac{\sin\frac{1}{2}(\theta+\theta')}{\cos\frac{1}{2}(\theta-\theta')};$$

and if $(\theta - \theta')$ is constant, plainly $x^2 + y^2$ is constant, and, therefore, the locus is a circle.

98. We shall next give one or two examples involving the problem of Art. 85, to find the co-ordinates of the points where a given line meets a given circle.

Ex. 1. *To find the locus of the middle points of chords of a given circle, drawn parallel to a given line.*

Let the equation of any of the parallel chords be

$$x \cos \alpha + y \sin \alpha - p = 0,$$

where α is, by hypothesis, given, and p is indeterminate; the abscissæ of the points where this line meets the circle are (Art. 85) found from the equation

$$x^2 - 2px \cos \alpha + p^2 - r^2 \sin^2 \alpha = 0.$$

Now, if the roots of this equation be x' and x'', the x of the middle point of the chord will (Art. 11) be $\frac{x' + x''}{2}$, or, from the theory of equations, will $= p \cos \alpha$. In like manner, the y of the middle point will equal $p \sin \alpha$. Hence the equation of the locus is $\frac{y}{x} = \tan \alpha$, that is, a right line drawn through the centre perpendicular to the system of parallel chords, since α is the angle made with the axis of x by a perpendicular to the chord

$$x \cos \alpha + y \sin \alpha - p = 0.$$

Ex. 2. *Given a line and a circle, to find a point such that if any chord be drawn through it, and perpendiculars let fall from its extremities on the given line, the rectangle under these perpendiculars will be constant.*

Take the given line for axis of y, and let the axis of x be the perpendicular on it from the centre of the given circle, whose length we shall call p. Then the equation of the circle is (Art. 82)

$$y^2 + (x - p)^2 = r^2.$$

Again, if the co-ordinates of the sought point be x', y', the equation of any line through it will be

$$(y - y') = m(x - x'), \text{ or } y = mx + y' - mx'.$$

Substitute this value of y in the equation of the circle, and we shall get, to determine the x of the points where the line meets the circle,

$$(1 + m^2) x^2 + \{2m(y' - mx') - 2p\} x + (y' - mx')^2 + p^2 - r^2 = 0.$$

But x is the perpendicular on the given line, and the product of the two perpendiculars (by the theory of equations)

$$= \frac{(y' - mx')^2 + p^2 - r^2}{1 + m^2}.$$

This will not be independent of m, unless the numerator be divisible by $1 + m^2$, and it will be found that this cannot be the case unless $y' = 0$ and $x'^2 = p^2 - r^2$. Hence there are two such points situated on the axis of x, and at a distance from the origin = the tangent drawn from it to the given circle.

Ex. 3. *If any chord be drawn through a fixed point on a diameter of a circle, and its extremities joined to either end of the diameter, the joining lines will cut off on the tangent, at the other end of the diameter, portions whose rectangle is constant.*

Let us take the diameter for axis of x, and either extremity of it for origin, then (Art. 82) the equation of the circle will be $x^2 + y^2 = 2rx$, and that of any chord through a fixed point on the diameter will be $y = m(x - x')$. By combining these equations we can determine the co-ordinates of the extremities of the chord. We can, however, without solving for these co-ordinates, obtain directly from the equations the equation of the lines joining these extremities to the origin. For if, by combining the equations, we can obtain a homogeneous function of the second degree, it will be, by Art. 72, the equation of two right lines drawn through the origin, and it evidently must be satisfied by the co-ordinates of the points which satisfy the two given equations.

The origin must be taken at that end of diam^r which is joined to ends of chd.

Write these equations thus,

$$x^2 + y^2 = 2rx, \text{ and } mx' = mx - y,$$

and, multiplying them together, we get

$$mx'(x^2 + y^2) = 2rx(mx - y).$$

This being homogeneous in x and y, is the required equation of the joining lines. It may be written thus,

$$mx' \cdot y^2 + 2r \cdot xy + m(x' - 2r)x^2 = 0.$$

This equation enables us to find the values of y corresponding to any value of x, and we see that the product of these values will be $\frac{x' - 2r}{x'}x^2$, and, therefore, independent of m. The intercepts

made on a perpendicular at the extremity of the diameter are found by making $x = 2r$ in the preceding equation, and their product is $4r^2 \frac{x' - 2r}{x'}$, which will be constant as long as x' is constant.

99. We shall next give one or two examples requiring the use of the equation of the polar of a point (found in Art. 89).

Ex. 1. *If any chord be drawn through a fixed point and tangents at its extremities: to find the locus of their point of intersection.*

Let any point on the locus be XY, then the chord joining points of contact of tangents passing through XY, is

$$\mathrm{X}x + \mathrm{Y}y = r^2;$$

but by hypothesis, this line passes through the point $x'y'$, therefore,

$$\mathrm{X}x' + \mathrm{Y}y' = r^2;$$

this is the relation connecting the co-ordinates of the point XY, its locus, therefore, is the line

$$xx' + yy' = r^2,$$

or the polar of the point $x'y'$.

The proposition just proved may be stated otherwise, thus:

If one point lie on the polar of a second point, the second point will lie on the polar of the first point.

For the equation of the polar of the second point is

$$xx'' + yy'' = r^2,$$

and the condition that $x'y'$ should lie on this line is

$$x'x'' + y'y'' = r^2.$$

But this is also the condition to be fulfilled, in order that the point $x''y''$ should lie on the line

$$xx' + yy' = r^2.$$

We reserve a more detailed consideration of the properties of poles and polars, until we come to treat of conic sections, as the reader is presumed to be familiar with the geometrical demonstration of these properties in the case of the circle.

Ex. 2. *Given any point* O, *and any two lines through it; join both directly and transversely the points in which these lines meet a circle; then, if the direct lines intersect each other in* P *and the trans-*

verse in Q, *the line* PQ *will be the polar of the point* O, *with regard to the circle.*

Take the two fixed lines for axes, and let the intercepts made on them by the circle be a and a', b and b'. Then

$$\frac{x}{a}+\frac{y}{b}-1=0, \quad \frac{x}{a'}+\frac{y}{b'}-1=0,$$

will be the equations of the direct lines; and

$$\frac{x}{a'}+\frac{y}{b}-1=0, \quad \frac{x}{a}+\frac{y}{b'}-1=0,$$

the equations of the transverse lines. Now, the equation of the line PQ will be

$$\frac{x}{a}+\frac{x}{a'}+\frac{y}{b}+\frac{y}{b'}-2=0,$$

(for, see Art. 48, this line passes through the intersection of

$$\frac{x}{a}+\frac{y}{b}-1, \quad \frac{x}{a'}+\frac{y}{b'}-1,$$

and also of

$$\frac{x}{a}+\frac{y}{b'}-1, \quad \frac{x}{a'}+\frac{y}{b}-1).$$

If the equation of the circle be

$$Ax^2+Bxy+Ay^2+Dx+Ey+F=0,$$

a and a' are determined from the equation $Ax^2+Dx+F=0$ (Art. 87), therefore,

$$\frac{1}{a}+\frac{1}{a'}=-\frac{D}{F}, \text{ and } \frac{1}{b}+\frac{1}{b'}=-\frac{E}{F}.$$

Hence, equation of PQ is

$$Dx+Ey+2F=0;$$

but we saw (Art. 90) that this was the equation of the polar of the origin O. Hence it appears, that if the point O were given, and the two lines through it were *not* fixed, the *locus* of the points P and Q would be the polar of the point O.

Ex. 3. *Given any two points* A *and* B, *and their polars, with respect to a circle whose centre is* O: *let fall a perpendicular* AP *from* A *on the polar of* B, *and a perpendicular* BQ *from* B *on the polar of* A; *then* $\frac{OA}{AP}=\frac{OB}{BQ}$.

The equation of the polar of $A(x'y')$ is $xx' + yy' - r^2 = 0$, if we take the centre of the circle for our origin; and BQ, the perpendicular on this line from $B(x''y'')$ is (Art. 31) $\dfrac{x'x'' + y'y'' - r^2}{\sqrt{(x'^2 + y'^2)}}$.

Hence, since $\sqrt{(x'^2 + y'^2)} = OA$, we find

$$OA \,.\, BQ = x'x'' + y'y'' - r^2,$$

and, for the same reason,

$$OB \,.\, AP = x'x'' + y'y'' - r^2,$$

Hence

$$\frac{OA}{AP} = \frac{OB}{BQ}.$$

Ex. 4. *Given a circle and a triangle* ABC, *if we take the polars with respect to the circle of the three vertices of the triangle, we shall form a new triangle* A′B′C′ (*where* A′ *is the pole of* BC, B′ *the pole of* AC, *and* C′ *the pole of* AB), *then the lines* AA′, BB′, CC′, *will all pass through the same point.*

The equation of the line joining the point $x'y'$ to the intersection of the two lines $xx'' + yy'' - r^2 = 0$ and $xx''' + yy''' - r^2 = 0$ is (Art. 49)

$$AA' \quad (x'x''' + y'y''' - r^2)\,(xx'' + yy'' - r^2) - (x'x'' + y'y'' - r^2)\,(xx''' + yy''' - r^2) = 0.$$

In like manner,

$$BB' \quad (x'x'' + y'y'' - r^2)\,(xx''' + yy''' - r^2) - (x''x''' + y''y''' - r^2)\,(xx' + yy' - r^2) = 0;$$

and

$$CC' \quad (x''x''' + y''y''' - r^2)\,(xx' + yy' - r^2) - (x'x''' + y'y''' - r^2)\,(xx'' + yy'' - r^2) = 0;$$

and by Art. 51 these lines must pass through the same point. From Art. 61 it appears that the three points of intersection of AB and A′B′, of AC and A′C′, and of BC and B′C′, will lie in one right line.

The following is a particular case of the present example. *If a circle be inscribed in a triangle, and each vertex of the triangle joined to the point of contact of the circle with the opposite side, the three joining lines will meet in a point.*

100. We shall conclude this chapter with some examples of the use of polar co-ordinates.

Ex. 1. We take for our first example the well-known theorem

(Euclid, III. 35, 36), that *if through a fixed point any chord of a circle be drawn, the rectangle under its segments will be constant.*

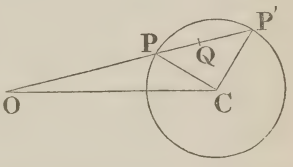
Take the fixed point for the pole, and this theorem is an immediate consequence from the form of the polar equation (Art. 94),

$$\rho^2 - 2\rho d\cos\theta + d^2 - r^2 = 0;$$

for the roots of this equation are evidently OP, OP′, the values of the radius vector answering to any given value of θ or POC.

Now, by the theory of equations, OP . OP′, the product of these roots will $= d^2 - r^2$, a quantity *independent of* θ, and therefore constant, whatever be the direction in which the line OP is drawn. If the point O be outside the circle, it is plain that $d^2 - r^2$ must be = the square of the tangent.

Ex. 2. *If through a fixed point* O *any chord of a circle be drawn, and* OQ *taken an arithmetic mean between the segments* OP, OP′; *to find the locus of* Q.



We can at once find the polar equation of the locus, that is to say, the relation between the radius vector OQ, and the angle θ or QOC. For OP + OP′, or the sum of the roots of the quadratic in the last example $= 2d\cos\theta$; but OP + OP′ = 2OQ, therefore, $OQ = d\cos\theta$. Hence the polar equation of the locus is

$$\rho = d\cos\theta.$$

Now it appears from the final equation in Art. 94, that this is the equation of a circle described on the line OC as diameter.

The question in this example might have been otherwise stated: "To find the locus of the middle points of chords which all pass through a fixed point."

Ex. 3. *If the line* OQ *had been taken an* harmonic *mean between* OP *and* OP′, *to find the locus of* Q.

That is to say, $OQ = \dfrac{2 . OP . OP'}{OP + OP'}$, but $OP . OP' = d^2 - r^2$, and $OP + OP' = 2d\cos\theta$, therefore, the polar equation of the locus is

$$\rho = \frac{d^2 - r^2}{d\cos\theta}, \text{ or } \rho\cos\theta = \frac{d^2 - r^2}{d}.$$

This is the equation of a right line (Art. 40) perpendicular to

OC, and at a distance from O = $d - \frac{r^2}{d}$, and, therefore, at a distance from C = $\frac{r^2}{d}$. Hence (Art. 90) the locus is the *polar* of the point O.

From Example 1 it appears, that if OQ had been taken a *geometric* mean between OP and OP′, the locus would be a circle whose centre would be O.

We can in like manner solve this and similar questions when the equation is given in the form

$$Ax^2 + Ay^2 + Dx + Ey + F = 0,$$

for, transforming to polar co-ordinates, the equation becomes

$$\rho^2 + \left(\frac{D}{A}\cos\theta + \frac{E}{A}\sin\theta\right)\rho + \frac{F}{A} = 0,$$

and, proceeding precisely as in this example, we find, for the locus of harmonic means,

$$\rho = \frac{-2F}{D\cos\theta + E\sin\theta},$$

and, returning to rectangular co-ordinates, the equation of the locus is

$$Dx + Ey + 2F = 0,$$

the same as the equation of the polar obtained already (Art. 90).

Ex. 4. *If on any radius vector through a fixed point* O, OQ *be taken in a constant ratio to* OP, *find the locus of* Q.

Let the equation of the given circle be

$$\rho^2 - 2\rho d\cos\theta + d^2 - r^2 = 0.$$

Then, since by hypothesis the radius vector OP = μ . OQ, we must substitute $\mu\rho$ for ρ, in order to find the equation of the locus, and we get

$$\rho^2 - 2\rho\frac{d}{\mu}\cos\theta + \frac{d^2 - r^2}{\mu^2} = 0.$$

Hence the locus is a circle having its centre on the line joining the point O to the centre of the given circle, and at a distance from O = $\frac{d}{\mu}$, while its radius = $\frac{r}{\mu}$.

If the equation of the first circle had been given, with regard to rectangular co-ordinates,

$$Ax^2 + Ay^2 + Dx + Ey + F = 0.$$

By transforming to polar co-ordinates it will be found, that the equation of the second is

$$\mu^2 . A(x^2 + y^2) + \mu . (Dx + Ey) + F = 0.$$

The point O has many important properties in relation to these two circles.

If any line drawn through it meet the first circle in the points P, P′, and the second in the points Q, Q′, then, by hypothesis, $OP = \mu OQ$, and $OP' = \mu OQ'$. From this property, that every line through O is cut similarly by the two circles, the point O is called the *centre of similitude* of the two circles.

A line drawn through O, to touch the first circle, must touch the second also, for in this case the points P and P′ will coincide, therefore OP = OP′, and, therefore, the proportionate quantities OQ and OQ′ must be equal, and, therefore, the points Q and Q′ will coincide, or (Art. 83) the line OQ will touch the second circle.

Since, then, the pair of tangents drawn through O to the first circle touch the second also, we learn that O, *the centre of similitude, is the point in which the common tangents to the circles intersect.*

Ex. 5. *If* OQ *be taken* inversely *as* OP, *find the locus of* Q.

This locus is obtained by substituting in the equation of the circle $\frac{k^2}{\rho}$ for ρ. This will give an equation exactly similar in form to that found in the last example, therefore the locus will be a circle, and the two circles will have the point O for their centre of similitude. Indeed, since, in the last example, the ratio $\frac{OP}{OQ}$, or $\frac{OP'}{OQ'}$, was constant, and since the rectangle OP . OP′ is constant (Ex. 1), it is evident that the rectangle OP . OQ′ = OQ . OP′ is constant.

Ex. 6. *Given a point and a right line; if* OQ *be taken inversely as* OP, *the radius vector to the right line, find the locus of* Q.

Ex. 7. *Given vertex and vertical angle of a triangle and rectangle under sides; if one base angle describe a right line or a circle, find locus described by the other base angle.*

Take vertex for pole; let the lengths of the sides be ρ and ρ',

and the angles they make with the axis θ and θ', then we have $\rho\rho' = k^2$ and $\theta - \theta' = C$.

The student must write down the polar equation of the locus which one base angle is said to describe; this will give him a relation between ρ and θ; then, writing for ρ, $\frac{k^2}{\rho'}$, and for θ, $C + \theta'$, he will find a relation between ρ' and θ', which will be the polar equation of the locus described by the other base angle.

This example might be solved in like manner, if the *ratio* of the sides, instead of their rectangle, had been given.

Ex. 8. *If through any point* O, *on the circumference of a circle, any three chords be drawn, and on each, as diameter, a circle be described, these three circles* (*which, of course, all pass through* O) *will intersect in three other points, which lie in one right line.**

Take the fixed point O for pole, then if d be the diameter of the original circle, its polar equation will be (Art. 94)

$$\rho = d \cos\theta.$$

In like manner, if the diameter of one of the other circles make an angle α with the fixed axis, its length will be $= d\cos\alpha$, and the equation of this circle will be

$$\rho = d\cos\alpha \,.\, \cos(\theta - \alpha).$$

The equation of another circle will, in like manner, be

$$\rho = d\cos\beta \,.\, \cos(\theta - \beta).$$

To find the polar co-ordinates of the point of intersection of these two, we should seek what value of θ would render

$$\cos\alpha \,.\, \cos(\theta - \alpha) = \cos\beta \,.\, \cos(\theta - \beta),$$

and it is easy to find that θ must $= \alpha + \beta$, and the corresponding value of $\rho = d\cos\alpha\cos\beta$.

Similarly, the polar co-ordinates of the intersection of the first and third circles are

$$\theta = \alpha + \gamma, \text{ and } \rho = d\cos\alpha\cos\gamma.$$

Now, to find the polar equation of the line joining these two points, take the general equation of a right line, $\rho\cos(k - \theta) = p$

* See *Cambridge Math. Jour.*, I. 169, for an ingenious solution of this question by a different method. The theorem itself was probably originally obtained by the method of deformation of curves to be explained afterwards.

(Art. 40), and substitute in it successively these values of θ and ρ, and we shall get two equations to determine p and k. We shall get

$$p = d\cos\alpha\cos\beta\cos(k - \overline{\alpha + \beta}) = d\cos\alpha\cos\gamma\cos(k - \overline{\alpha + \gamma}).$$

Hence $$k = \alpha + \beta + \gamma, \text{ and } p = d\cos\alpha\cos\beta\cos\gamma.$$

The symmetry of those values shows that it is the same right line which joins the intersections of the first and second, and of the second and third circles, and, therefore, that the three points are in a right line.

CHAPTER VIII.

THE CIRCLE, CONTINUED.

101. In the last two chapters we investigated chiefly properties which only required the consideration of one circle, and in these we simplified our calculations by a particular choice of the axes of co-ordinates; in the present chapter we purpose, first, to discuss some properties of a single circle (the axes having any position), and then to show how to obtain analytically the properties of a system of two or more circles.

To find the equation of a circle passing through three given points.

We shall first suppose one of the given points the origin, and having found the equation for this case, we can obtain the equation for the more general case by writing $x - x_1$ for x, and $y - y_1$ for y. The equation of any circle through the origin is (Art. 82)

$$x^2 + y^2 + Dx + Ey = 0;$$

and we determine D and E by substituting in it the co-ordinates of the two given points. Writing ρ^2 for $x^2 + y^2$, this gives us

$$\rho_2^2 + Dx_2 + Ey_2 = 0,$$
$$\rho_3^2 + Dx_3 + Ey_3 = 0.$$

From these equations D and E are determined, and then writing $x - x_1$ for x, and $y - y_1$ for y, the equation of a circle through the three points x_1y_1, x_2y_2, x_3y_3, will be found to be

$$\begin{aligned}&(x^2+y^2)\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}\\&-(x_1^2+y_1^2)\{x_2(y_3-y)+x_3(y-y_2)+x(y_2-y_3)\}\\&+(x_2^2+y_2^2)\{x_3(y-y_1)+x(y_1-y_3)+x_1(y_3-y)\}\\&-(x_3^2+y_3^2)\{x(y_1-y_2)+x_1(y_2-y)+x_2(y-y_1)\}=0.\end{aligned}$$

If it were required to find the condition that four points should lie on the circumference of the same circle, we have only to write the co-ordinates x_4y_4 for x and y in this equation.

From the resulting equation it appears that if A, B, C, D be any four points on a circle, and O any fifth point taken arbitrarily,

$$OA^2.BCD + OC^2.ABD = OB^2.ACD + OD^2.ABC,$$

denoting by BCD the area of the triangle BCD, &c.

102. In the last article we obtained the equation of the circle circumscribing a triangle in terms of the co-ordinates of its angles; it is, however, often more convenient in practice to express it in terms of the equations of its sides, as follows:

Let the equations of the three sides be $\alpha=0$, $\beta=0$, $\gamma=0$, then any equation of the form

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

is the equation of a curve of the second degree passing through the three given points, since it is satisfied by supposing

$$\alpha=0\ \beta=0, \quad \alpha=0\ \gamma=0, \quad \text{or} \quad \beta=0\ \gamma=0.$$

We must then (as in Art. 95) find what the values of $\frac{m}{l}$ and $\frac{n}{l}$ must be, in order that this should be the equation of a circle. Writing for α, &c., at full length, $x\cos\alpha + y\sin\alpha - p$, and developing the equation, we obtain the two following conditions,

$$\begin{aligned}&l\cos(\beta+\gamma)+m\cos(\gamma+\alpha)+n\cos(\alpha+\beta)=0,\\&l\sin(\beta+\gamma)+m\sin(\gamma+\alpha)+n\sin(\alpha+\beta)=0.\end{aligned}$$

Solving these equations we obtain

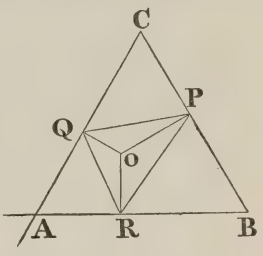
$$\frac{m}{l}=\frac{\sin(\gamma-\alpha)}{\sin(\beta-\gamma)}, \qquad \frac{n}{l}=\frac{\sin(\alpha-\beta)}{\sin(\beta-\gamma)}.$$

Now if C be the angle contained by the sides α, β, then sin C = sin $(\alpha - \beta)$, &c. (since $\alpha - \beta$) is the angle between the

perpendiculars on those sides), hence the equation of the circle circumscribing a triangle is,

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0.$$

103. The geometrical interpretation of the equation just found deserves attention. If from any point O we let fall perpendiculars OP, OQ, on the lines α, β, then (Art. 53) α, β, are the lengths of these perpendiculars; and since the angle between them is the supplement of C, the quantity $\alpha\beta \sin C$ is double the area of the triangle OPQ. In like manner, $\alpha\gamma \sin B$ and $\beta\gamma \sin A$ are double the triangles OPR, OQR. Hence the quantity



$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C$$

is double the area of the triangle PQR, and the equation found in the last Article asserts, that if the point O be taken on the circumference of the circumscribing circle, the area PQR will vanish, that is to say (Art. 35), the three points P, Q, R will lie on one right line.

If it were required to find the locus of a point from which, if we let fall perpendiculars on the sides of a triangle, and join their feet, the triangle PQR so formed should have a constant magnitude, the equation of the locus would be

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = \text{const.}$$

and, since this only differs from the equation of the circumscribing circle in the constant part, it is (Art. 81) the equation of a circle concentric with the circumscribing circle.

104. From the equation

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0,$$

we can find the equations of the tangents to the circle at the vertices of the triangle. Put the equation into the form

$$\gamma(\beta \sin A + \alpha \sin B) + \alpha\beta \sin C = 0,$$

and we saw (in Art. 102) that γ meets the circle in the two points where it meets the lines α and β, since, if we make $\gamma = 0$ in the equation of the circle, that equation will be reduced to $\alpha\beta = 0$.

Now, for the very same reason, the two points in which the line $\beta \sin A + \alpha \sin B$ meets the circle, are the two points where *it* meets the lines α and β. But these two points coincide, since $\beta \sin A + \alpha \sin B$ passes through the point $\alpha\beta$. Hence, since the line $\beta \sin A + \alpha \sin B$ meets the circle in two coincident points, it is (Art. 83) a tangent at the point $\alpha\beta$.

We saw (Art. 63) that $\alpha \sin A + \beta \sin B$ is the equation of a parallel to the base (γ) drawn through the vertex $\alpha\beta$. Hence, by Art. 58, the tangent $\alpha \sin B + \beta \sin A$ makes the same angle with one side, that the base makes with the other (Euclid, III. 32).

From the forms of the equations of the three tangents,

$$\frac{\alpha}{\sin A} + \frac{\beta}{\sin B} = 0, \quad \frac{\beta}{\sin B} + \frac{\gamma}{\sin C} = 0, \quad \frac{\gamma}{\sin C} + \frac{\alpha}{\sin A} = 0,$$

it appears, that the three points in which they intersect each the opposite side are in one right line, whose equation is

$$\frac{\alpha}{\sin A} + \frac{\beta}{\sin B} + \frac{\gamma}{\sin C} = 0.$$

It will be found that the equations of the lines joining the vertices of the inscribed triangle to those of the circumscribed, are,

$$\frac{\alpha}{\sin A} - \frac{\beta}{\sin B} = 0, \quad \frac{\beta}{\sin B} - \frac{\gamma}{\sin C} = 0, \quad \frac{\gamma}{\sin C} - \frac{\alpha}{\sin A} = 0;$$

and these meet in a point, the equations being of the form discussed in Art. 60.

105. We shall next show how to obtain the equation of the circle *inscribed* in the triangle α, β, γ. The equation

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0,$$

represents a curve of the second degree, inscribed in the triangle $\alpha\beta\gamma$, for if we seek the point where any side (γ) cuts the figure, making $\gamma = 0$, we obtain the perfect square,

$$l^2\alpha^2 + m^2\beta^2 - 2lm\alpha\beta = 0;$$

the roots of this equation being equal, we infer that the two points coincide in which γ cuts the figure, and therefore (Art. 83) that γ is a tangent.

In the same manner it can be proved that the sides α and β touch the curve represented by the preceding equation.

This equation may also be written in a convenient form,

$$l^{\frac{1}{2}}a^{\frac{1}{2}} + m^{\frac{1}{2}}\beta^{\frac{1}{2}} + n^{\frac{1}{2}}\gamma^{\frac{1}{2}} = 0;$$

for if we clear this equation of radicals we shall find it to be identical with that just written.

For the simplest method of obtaining the particular values of l, m, n, for which the preceding equation represents *a circle*, I am indebted to Dr. Hart, who derives the equation of the inscribed circle from that of the circumscribed, as follows: Join the points of contact of the circle inscribed in a triangle; let the equations of the sides of the triangle so formed be $\alpha' = 0$, $\beta' = 0$, $\gamma' = 0$, and its angles A′, B′, C′; then (Art. 100) the equation of the circle must be

$$\beta'\gamma' \sin A' + \gamma'\alpha' \sin B' + \alpha'\beta' \sin C' = 0.$$

Now we have proved (Art. 93, Ex. 1) that for any point on the circle

$$\alpha'^2 = \beta\gamma; \quad \beta'^2 = \gamma\alpha; \quad \gamma'^2 = \alpha\beta,$$

and it is easy to see that

$$A' = 90° - \tfrac{1}{2}A; \quad B' = 90° - \tfrac{1}{2}B; \quad C' = 90° - \tfrac{1}{2}C.$$

Substituting these values, the equation of the circle becomes

$$\alpha^{\frac{1}{2}} \cos \tfrac{1}{2}A + \beta^{\frac{1}{2}} \cos \tfrac{1}{2}B + \gamma^{\frac{1}{2}} \cos \tfrac{1}{2}C = 0.$$

The general equation will, therefore, represent a circle if l, m, n, be proportional to $\cos^2 \tfrac{1}{2}A$, $\cos^2 \tfrac{1}{2}B$, $\cos^2 \tfrac{1}{2}C$.

It can be proved, in like manner, that the equation of the circle touching the side a, and the sides b and c *produced*, is

$$\alpha^{\frac{1}{2}} \cos \tfrac{1}{2}A + \beta^{\frac{1}{2}} \sin \tfrac{1}{2}B + \gamma^{\frac{1}{2}} \sin \tfrac{1}{2}C = 0.$$

106. In the last article we saw that the general equation might be written in the form

$$n\gamma\,(n\gamma - 2l\alpha - 2m\beta) + (l\alpha - m\beta)^2 = 0,$$

and hence that the line $(l\alpha - m\beta)$, which obviously passes through the point $\alpha\beta$, passes also through the point where γ meets the curve. The three lines, then, which join the points of contact of the sides with the opposite angles of the circumscribing triangle, are

$$l\alpha - m\beta = 0, \quad m\beta - n\gamma = 0, \quad n\gamma - l\alpha = 0,$$

and these obviously meet in a point.

The very same proof which showed that γ touches the curve

shows also that $n\gamma - 2l\alpha - 2m\beta$ touches the curve, for when this quantity is put = 0, we have the perfect square $(l\alpha - m\beta)^2 = 0$; hence this line meets the curve in two coincident points, that is, touches the curve, and $l\alpha - m\beta$ passes through the point of contact. Hence, if the vertices of the triangle be joined to the points of contact of opposite sides, and at the points where the joining lines meet the circle again, tangents be drawn, their equations are

$$2l\alpha + 2m\beta - n\gamma = 0,\ 2m\beta + 2n\gamma - l\alpha = 0,\ 2n\gamma + 2l\alpha - m\beta = 0.$$

Hence we infer that the three points, where each of these tangents meets the opposite side, lie in one right line,

$$l\alpha + m\beta + n\gamma = 0,$$

for this line passes through the intersection of the first line with γ, of the second with α, and of the third with β.

107. We now proceed to examine the properties of *two or more* circles, and commence with the question: "*To find the equation of the chord of intersection of two circles.*"

If $S = 0$, $S' = 0$, be the equations of two circles, then any equation of the form $S - kS' = 0$ will be the equation of a figure passing through their points of intersection (Art. 48).

Let us write down the equations

$$S = (x - a)^2 + (y - b)^2 - r^2 = 0,$$
$$S' = (x - a')^2 + (y - b')^2 - r'^2 = 0,$$

and it is evident that the equation $S - kS' = 0$, will in general represent a circle, since the co-efficient of $xy = 0$, and that of x^2 = that of y^2. There is one case, however, where it will represent a right line, namely, when $k = 1$. The terms of the second degree then vanish, and the equation becomes

$$S - S' = 2(a' - a)x + 2(b' - b)y + r'^2 - r^2 + a^2 - a'^2 + b^2 - b'^2 = 0.$$

This is, therefore, the equation of the right line passing through the points of intersection of the two circles.*

108. The points of intersection of the two circles are found by seeking, as in Art. 85, the points in which the line $S - S'$

* We shall have occasion, in the next Part, to examine more closely the nature of the chords of intersection of two circles.

meets either of the given circles. These points will be real, coincident, or imaginary, according to the nature of the roots of the resulting equation; but it is remarkable that, whether the circles meet in real or imaginary points, the equation of the chord of intersection, $S - S' = 0$, always represents a real line, having important geometrical properties in relation to the two circles. This confirms our assertion (Art. 87), that the line joining two points preserves its existence and its properties when those points have become imaginary.

In order to avoid the harshness of calling the line $S - S' = 0$ the chord of intersection in the case where the circles do not *geometrically* appear to intersect, it has been called the *radical axis* of the two circles.

109. One of the most remarkable properties of this line is found by examining the geometric meaning of the equation $S - S' = 0$. We saw (Art. 93) that if the co-ordinates of any point xy be substituted in the quantity S, it represents the square of the tangent drawn to the circle S, from the point xy. So also S′ is the square of the tangent drawn to the circle S′, and the equation $S - S' = 0$ asserts, that *if from any point on the radical axis tangents be drawn to the two circles, these tangents will be equal.*

The line $(S - S')$ possesses this property whether the circles meet in real points or not. When the circles do not meet in real points, the position of the radical axis is determined geometrically by cutting the line joining their centres, so that the difference of the squares of the parts may = the difference of the squares of the radii, and erecting a perpendicular at this point; as is evident, since the tangents from this point must be equal to each other.

If it were required to find the locus of a point whence tangents to two circles have *a given ratio*, it appears, from Art. 93, that the equation of the locus will be,

$$S - k^2 S' = 0,$$

which (Art. 107) represents a circle passing through the real or imaginary points of intersection of S and S′. When the circles S and S′ do not intersect in real points, we may express the rela-

tion which they bear to the circle $S - k^2S'$ by saying that the three circles have a common radical axis.

110. From the form of the equation of the radical axis of two circles we at once derive the following theorem.

Given any three circles, if we take the radical axis of each pair of circles, these three lines will meet in a point, and this point is called the radical centre *of the three circles.*

For the equations of the three radical axes are,

$$S - S' = 0, \quad S' - S'' = 0, \quad S'' - S = 0,$$

which, by Art. 51, meet in a point.

From this theorem we immediately derive the following:

If several circles pass through two fixed points, their chord of intersection with a fixed circle will pass through a fixed point.

For, imagine one circle through the two given points to be fixed, then its chord of intersection with the given circle will be fixed; and its chord of intersection with any variable circle drawn through the given points, will plainly be the fixed line joining the two given points. These two lines determine, by their intersection, a fixed point through which the chord of intersection of the variable circle with the first given circle must pass.

111. A system of circles having a common radical axis possesses many remarkable properties which are more easily investigated by taking the radical axis for the axis of y, and the line joining the centres for the axis of x. Then the equation of any circle will be

$$x^2 + y^2 - 2kx + \delta^2 = 0,$$

where δ^2 is the same for all the circles of the system, and the equations of the different circles are obtained by giving different values to k.

For it is evident (Art. 82) that the centre is on the axis of x, at the variable distance k, and if we take any two circles,

$$x^2 + y^2 - 2k'x + \delta^2 = 0,$$
$$x^2 + y^2 - 2k''x + \delta^2 = 0,$$

and subtract one equation from the other, their chord of intersection will be $x = 0$, or the axis of y.

When we give to δ^2 the sign $+$, the radical axis will meet the circles in imaginary points, and when we give the sign $-$, in real points.

112. *If several circles pass through two fixed points, the polar of a given point, with regard to any of them, will always pass through a fixed point.*

The equation of the circle may be written

$$y^2 + (x - k)^2 = k^2 - \delta^2,$$

therefore (Art. 90), equation of polar will be

$$yy' + (x - k)\,(x' - k) = k^2 + \delta^2,$$

or
$$yy' + xx' + \delta^2 - k\,.\,(x + x') = 0;$$

therefore, since this line involves the indeterminate k in the first degree (Art. 69), this line will always pass through the intersection of $yy' + xx' + \delta^2 = 0$, and $x + x' = 0$.

113. *There can always be found two points, however, such that their polars, with regard to any of the circles, will not only pass through a fixed point, but will be altogether fixed.*

This will happen when $yy' + xx' + \delta^2 = 0$, and $x + x' = 0$, represent the same right line, for this right line would then be the polar whatever the value of k. But that this should be the case we must have

$$y' = 0 \text{ and } x'^2 = \delta^2, \text{ or } x' = \pm\,\delta.$$

The two points whose co-ordinates have been just found have many remarkable properties in the theory of these circles, and are such that the polar of either of them, with regard to any of the circles, is a line drawn through the other perpendicular to the line of centres.

It is evident, that the equation

$$y^2 + (x - k)^2 = k^2 - \delta^2$$

(Art. 112) cannot represent a real circle if k^2 be less than δ^2; and if $k^2 = \delta^2$, then the equation will be of Class V. (Art. 82), and will represent a circle of infinitely small radius, the co-ordinates of whose centre are $y = 0$, $x = \pm\,\delta$. Hence the points just found may themselves be considered as circles of the system, and have, accordingly, been termed by Poncelet* the *limiting* points of the system of circles.

114. If from any point on the radical axis we draw tangents

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* *Traité des Propriétés projectives*, p. 41.

to all these circles, the locus of the points of contact must be a circle, since we proved (Art. 109) that all these tangents were equal. The equation of this circle can be readily found.

The square of the tangent from any point $(x = 0, y = h)$ to the circle

$$x^2 + y^2 - 2kx + \delta^2 = 0,$$

being found by substituting these co-ordinates in this equation, $= h^2 + \delta^2$; and the circle whose centre is the point $(x = 0, y = h)$, and whose radius squared $= h^2 + \delta^2$, must have for its equation

$$x^2 + (y - h)^2 = h^2 + \delta^2,$$

or

$$x^2 + y^2 - 2hy = \delta^2.$$

Hence, whatever be the point taken on the radical axis (*i. e.*, whatever the value of h may be), still this circle will always pass through the fixed points $(y = 0, x = \pm \delta)$ found in the last Article.

115. *To draw a common tangent to two circles.*

Let their equations be

$$(x - a)^2 + (y - b)^2 = r^2 \qquad \text{(S)},$$

and

$$(x - a')^2 + (y - b')^2 = r'^2 \qquad \text{(S')}.$$

We saw (Art. 84) that the equation of a tangent to (S) was

$$(x - a)(x' - a) + (y - b)(y' - b) = r^2;$$

or, as in Art. 97, writing

$$\frac{x' - a}{r} = \cos\alpha, \quad \frac{y' - b}{r} = \sin\alpha,$$

$$(x - a)\cos\alpha + (y - b)\sin\alpha = r.$$

In like manner, any tangent to (S') is

$$(x - a')\cos\beta + (y - b')\sin\beta = r'.$$

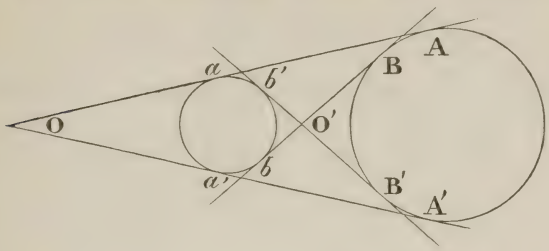
Now, if we seek the conditions necessary that these two equations should represent *the same* right line; first, from comparing the ratio of the co-efficients of x and y, we get $\tan\alpha = \tan\beta$, whence β either $= \alpha$, or $= 180° + \alpha$. If either of these conditions be fulfilled, we must equate the absolute terms, and we find, in the first case,

$$(a - a')\cos\alpha + (b - b')\sin\alpha + r - r' = 0,$$

and in the second case,

$$(a - a')\cos\alpha + (b - b')\sin\alpha + r + r' = 0.$$

Either of these equations would give us a quadratic to determine a. The two roots of the first equation would correspond to the



direct or exterior common tangents, Aa, A′a′; the roots of the second equation would correspond to the transverse or interior tangents, Bb, B′b′.

If we wished to find the co-ordinates of the point of contact of the common tangent with the circle (S), we must substitute, in the equation just found, for $\cos a$, its value, $\frac{x' - a}{r}$, and for $\sin a$, $\frac{y' - b}{r}$, and we find

$$(a - a')\,(x' - a) + (b - b')\,(y' - b) + r(r - r') = 0;$$

or else,

$$(a - a')\,(x' - a) + (b - b')\,(y' - b) + r(r + r') = 0.$$

The first of these equations, combined with the equation (S) of the circle, will give a quadratic, whose roots will be the co-ordinates of the points A and A′, in which the direct common tangents touch the circle (S); and it will appear, as in Art. 89, that

$$(a' - a)\,(x - a) + (b' - b)\,(y - b) = r(r - r')$$

is the equation of AA′, the chord of contact of direct common tangents. So, likewise,

$$(a' - a)\,(x - a) + (b' - b)\,(y - b) = r(r + r')$$

is the equation of the chord of contact of transverse common tangents. If the origin be the centre of the circle (S), then a and $b = 0$; and we find, for the equation of the chord of contact,

$$a'x + b'y = r\,(r \mp r').$$

116. The points O and O′, in which the direct or transverse tangents intersect, are called the *centres of similitude* of the two circles (see p. 99).

Their co-ordinates are easily found, for O is the pole, with regard to circle (S), of the chord AA′, whose equation is

$$\frac{(a'-a)\,r}{r-r'}(x-a)+\frac{(b'-b)\,r}{r-r'}(y-b)=r^2.$$

Comparing this equation with the equation of the polar of the point $x'y'$,

$$(x'-a)(x-a)+(y'-b)(y-b)=r^2,$$

we get

$$x'-a=\frac{(a'-a)\,r}{r-r'},\quad \text{or } x'=\frac{a'r-ar'}{r-r'},$$

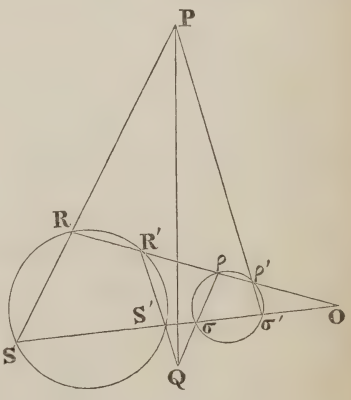
$$y'-b=\frac{(b'-b)\,r}{r-r'},\quad \text{or } y'=\frac{b'r-br'}{r-r'}.$$

So, likewise, the co-ordinates of O′ are found to be

$$x=\frac{a'r+ar'}{r+r'},\quad \text{and } y=\frac{b'r+br'}{r+r'}.$$

These values of the co-ordinates indicate (see Art. 11) that the centres of similitude are the points where the line joining the centres is cut externally and internally in the ratio of the radii.

117. *If through a centre of similitude we draw any two lines meeting the first circle in the points* R, R′, S, S′, *and the second in the points* ρ, ρ′, σ, σ′, *then the chords* RS, ρσ; R′S′, ρ′σ′; *will be parallel, and the chords* RS, ρ′σ′; R′S′, ρσ; *will meet on the radical axis of the two circles.*



Take the two fixed lines for axes, then we saw (Art. 100, Ex. 4) that $OR=\mu O\rho$, $OS=\mu O\sigma$, and that if the equation of the circle $\rho\sigma\rho'\sigma'$ be

$$Ax^2+Bxy+Ay^2+Dx+Ey+F=0,$$

that of the other will be

$$Ax^2+Bxy+Ay^2+\mu.(Dx+Ey)+\mu^2F=0,$$

and, therefore, the equation of the radical axis will be (Art. 108)

$$Dx+Ey+(\mu+1)F=0.$$

Now let the equation of $\rho\sigma$ be

$$\frac{x}{a} + \frac{y}{b} = 1,$$

and that of $\rho'\sigma'$,

$$\frac{x}{a'} + \frac{y}{b'} = 1,$$

then the equation of RS must be

$$\frac{x}{\mu a} + \frac{y}{\mu b} = 1,$$

and that of R′S′,

$$\frac{x}{\mu a'} + \frac{y}{\mu b'} = 1.$$

It is evident, from the form of the equations, that RS is parallel to $\rho\sigma$; and, as in Art. 99, Ex. 2, RS and $\rho'\sigma'$ must intersect on the line

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) + y\left(\frac{1}{b} + \frac{1}{b'}\right) = 1 + \mu,$$

or on

$$Dx + Ey + (1 + \mu)F = 0,$$

the radical axis of the two circles.

A particular case of this theorem is, that the tangents at R and ρ are parallel, and that the tangents at R and ρ' meet on the radical axis.

118. *Given three circles; the line joining a centre of similitude of the first and second to a centre of similitude of the first and third, will pass through a centre of similitude of the second and third.*

Form the equation of the line joining the points

$$x = \frac{ra' - ar'}{r - r'}, \quad y = \frac{rb' - br'}{r - r'},$$

and

$$x = \frac{ra'' - ar''}{r - r''}, \quad y = \frac{rb'' - br''}{r - r''},$$

(Art. 116), and we get

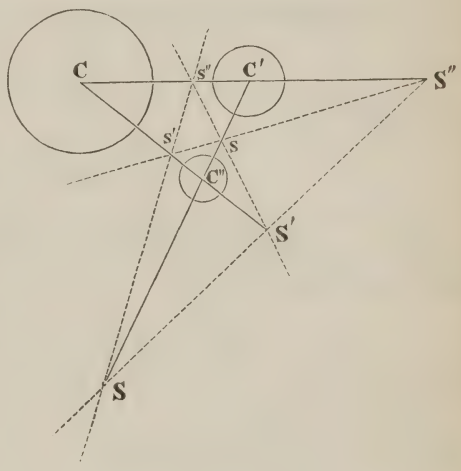
$$\begin{aligned} &\{r(b' - b'') + r'(b'' - b) + r''(b - b')\}x \\ &- \{r(a' - a'') + r'(a'' - a) + r''(a - a')\}y \\ &+ r(b'a'' - b''a') + r'(b''a - ba'') + r''(ba' - b'a) = 0. \end{aligned}$$

Now the symmetry of this equation sufficiently shows, that the line it represents must pass through the third centre of similitude,

$$x = \frac{r'a'' - r''a'}{r' - r''}, \quad y = \frac{r'b'' - r''b'}{r' - r''}.$$

This line is called an *axis of similitude* of the three circles.

Since for each pair of circles there are two centres of similitude, there will be in all *six* for the three circles, and these will be distributed along *four* axes of similitude, as represented in the figure. The equations of the other three will be found by changing the signs of r and r', or of r' and r'', or of r and r'', in the equation just given.



119. One particular case of the preceding theorem is often useful.

If a circle (Σ) *touch two others* (S *and* S′) *the line joining the points of contact will pass through a centre of similitude of* S *and* S′.

For when two circles touch, one of their centres of similitude will coincide with the point of contact, and, by the theorem proved in the last article, the line joining a centre of similitude of S and Σ, to a centre of similitude of (S′ and Σ) must pass through a centre of similitude of S and S′.

If Σ touch S and S′, either both externally or both internally, the line joining the points of contact will pass through the *external* centre of similitude of S and S′. If Σ touch one externally and the other internally, the line joining the points of contact will pass through the *internal* centre of similitude.

120. We shall conclude this chapter by investigating the problem : *To describe a circle to touch three given circles.*

Let the equations of the three circles be

$$(x-a)^2+(y-b)^2-r^2=0, \text{ or } S=0,$$
$$(x-a')^2+(y-b')^2-r'^2=0, \text{ or } S'=0,$$
$$(x-a'')^2+(y-b'')^2-r''^2=0, \text{ or } S''=0.$$

We can determine the position of the centre of the touching circle from the condition, that the distance between the centres of any two touching circles must equal the sum of their radii.

Now the square of the distance of any point from the centre of (S) $= (x-a)^2+(y-b)^2 = S+r^2$.

Hence we get the condition

$$S+r^2=(R+r)^2,$$

and, in like manner,

$$S'+r'^2=(R+r')^2,$$

and

$$S''+r''^2=(R+r'')^2.$$

(These equations, evidently, apply to the case of *external* contact. If the contact with any of the circles be *internal*, the distance between the centres will then = the *difference* of the radii, and we must change the sign of r or r' or r'' in the preceding formulæ).

Hence it appears, that there may be *eight* circles touching the three given circles, according to the different signs which we may give to r, r', r'', *e. g.*,

r, $+\,+\,+\,+\,-\,-\,-\,-$,
r', $+\,+\,-\,-\,+\,+\,-\,-$,
r'', $+\,-\,+\,-\,+\,-\,+\,-$.

If we eliminate R from the preceding formulæ, we shall get two equations which will enable us to determine the co-ordinates of the centre of the touching circle.

Subtract the equations, and we get

$$S-S'=2R(r-r'), \text{ and } S-S''=2R(r-r''),$$

or

$$\frac{S-S'}{r-r'}=\frac{S-S''}{r-r''}.$$

This is the equation of the line joining the centre of the touching circle to the *radical centre* (Art. 110). It may be written in the more symmetrical form

$$(r'-r'')S+(r''-r)S'+(r-r')S''=0.$$

If we now write for S, &c., their values, the co-efficient of x in this equation is found to be

$$-2\{a(r'-r'')+a'(r''-r)+a''(r-r')\},$$

and of y to be

$$-2\{b(r'-r'')+b'(r''-r)+b''(r-r')\}.$$

Now if we compare these co-efficients with the coefficients in the equation of the axis of similitude (Art. 118), we arrive at the conclusion (see Art. 38) that *the centre of the circle touching three others lies on the perpendicular let fall from their radical centre on the axis of similitude.*

We saw that eight circles can be drawn to touch three given circles, and as the three circles have four axes of similitude, the centres of the touching circles will lie, a pair on each of the perpendiculars let fall from the radical centre on the four axes of similitude.

Two circles answer to each axis of similitude; for the equation of an axis of similitude (Art. 118) remains unaltered, if we change in it the signs of *all* the radii. Hence the axis answering to the case of external contact (or $+r+r'+r''$) must also answer to the case of internal contact (or $-r-r'-r''$); and similarly for the other axes of similitude.

121. From the three equations found in the last Article we can obtain another relation between the co-ordinates of the centre of the touching circle. This relation, however, will be of the second degree, and, though sufficient for the *algebraical* solution of the problem, does not enable us to represent the results in an *elementary geometrical* manner. To remove this inconvenience M. Gergonne proposed to seek the co-ordinates not of the *centre* of the touching circle, but of its *point of contact* with one of the given circles. We have already one relation connecting these co-ordinates, since the point lies on a given circle; therefore, if we can find another relation between them, it will suffice completely to determine the point.*

Let us for simplicity take for origin the centre of the circle, the point of contact with which we are seeking, and let the equations of the three circles be

* Gergonne, *Annales des Mathématiques*, vol. vii. p. 289.

$$(x - a)^2 + (y - b)^2 = r^2 \qquad (S),$$
$$(x - a')^2 + (y - b')^2 = r'^2 \qquad (S'),$$
$$x^2 + y^2 = r''^2 \qquad (S''),$$

and of the sought circle,

$$(x - A)^2 + (y - B)^2 = R^2 \quad (\Sigma);$$

then, as in Art. 120, we get the three relations,

$$A^2 + B^2 = (R + r'')^2,$$
$$(A - a')^2 + (B - b')^2 = (R + r')^2$$
$$(A - a)^2 + (B - b)^2 = (R + r)^2,$$

whence we get

$$2Aa' + 2Bb' = a'^2 + b'^2 + r''^2 - r'^2 + 2(r'' - r')R,$$

and

$$2Aa + 2Bb = a^2 + b^2 + r''^2 - r^2 + 2(r'' - r)R.$$

Now it is proposed, instead of determining A, B, and R, from these equations, to seek the co-ordinates of the point of contact of the circles (S″) and (Σ). If x and y be these co-ordinates, we have, from similar triangles,

$$A = \frac{x(R + r'')}{r''}, \qquad B = \frac{y(R + r'')}{r''}.$$

Substitute these values in the last found equations, and we find

$$\{2ax + 2by + 2(r - r'')r''\}\frac{R + r''}{r''} = a^2 + b^2 - (r - r'')^2,$$

$$\{2a'x + 2b'y + 2(r' - r'')r''\}\frac{R + r''}{r''} = a'^2 + b'^2 - (r' - r'')^2.$$

Hence

$$\frac{ax + by + (r - r'')r''}{a^2 + b^2 - (r - r'')^2} = \frac{a'x + b'y + (r' - r'')r''}{a'^2 + b'^2 - (r' - r'')^2}.$$

This is the equation of a right line, whose intersection with the circle (S″) determines the points of contact required.

122. Although what has been said is sufficient for the algebraical solution of the problem, yet, in order to complete the geometrical solution, it is necessary to show how to construct the line whose equation has been just found. This will be done if we can find any two points on the line. But the form of the equation at once points to one point on the line; for (Art. 48) it must pass through the intersection of the lines

$$ax + by + (r - r'')\,r'' = 0,$$

and
$$a'x + b'y + (r' - r'')\,r'' = 0;$$

but (Art. 115) the line

$$ax + by + (r - r'')\,r'' = 0$$

is the chord of contact of common tangents to the circles (S) and (S″), or, in other words (Art. 116), is the polar, with regard to the circle S″, of the centre of similitude of these circles. In like manner,

$$a'x + b'y + (r' - r'')\,r'' = 0$$

is the polar of the centre of similitude of the circles (S) and (S″); therefore (since the intersection of any two lines is the pole of the line joining their polars), the intersection of the lines

$$ax + by + (r - r'')\,r'', \text{ and } a'x + b'y + (r' - r'')\,r'',$$

is the pole of the axis of similitude of the three circles, with regard to the circle (S″).

Again, put the last equation of the last Article into the form

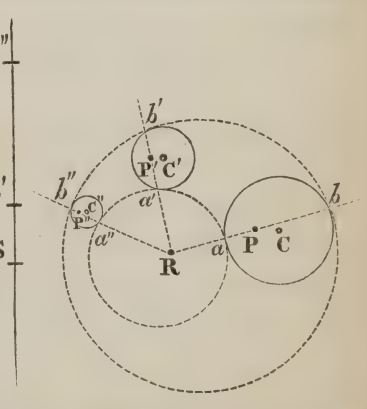
$$\frac{2ax + 2by + 2(r - r'')\,r''}{a^2 + b^2 - (r - r'')^2} - 1 = \frac{2a'x + 2b'y + 2(r' - r'')\,r''}{a'^2 + b'^2 - (r' - r'')^2} - 1,$$

or

$$\frac{2ax + 2by - a^2 - b^2 + r^2 - r''^2}{a^2 + b^2 - (r - r'')^2} = \frac{2a'x + 2b'y - a'^2 - b'^2 + r'^2 - r''^2}{a'^2 + b'^2 - (r' - r'')^2};$$

but
$$2ax + 2by - a^2 - b^2 + r^2 - r''^2 = 0$$

is (Art. 107) the equation of the radical axis of the circles (S) and (S″), and the numerator of the right hand member of the equation represents the radical axis of the circles S′ and S″; hence the line represented by the equation will pass also through the *radical centre* of the three circles.



Hence we obtain the following construction:

Drawing any of the four axes of similitude of the three circles, take its pole with respect to each circle, and join the points so found (P, P′, P″) with the radical centre (Art. 110); then, if the joining lines meet

the circles in the points $(a, b;\ a', b';\ a'', b'')$, the circle through a, a', a'' will be one of the touching circles, and that through b, b', b'' will be another. Repeating this process with the other three axes of similitude, we can determine the other six touching circles.

123. It is useful to show how the preceding results may be derived without algebraical calculations.

(1). By Art. 119 the lines ab, $a'b'$, $a''b''$ meet in a point, viz., the centre of similitude of the circles $aa'a''$, $bb'b''$.

(2). In like manner $a'a''$, $b'b''$ intersect in S, the centre of similitude of C', C''.

(3). Hence (Art. 117) the transverse lines $a'b'$, $a''b''$ intersect in the radical axis of C', C''. So again $a''b''$, ab, intersect on the radical axis of C'', C. Therefore the point R (the centre of similitude of $aa'a''$, $bb'b''$) must be the radical centre of the circles C, C', C''.

(4). In like manner, since $a'b'$, $a''b''$ pass through a centre of similitude of $aa'a''$, $bb'b''$; therefore (Art. 117) $a'a''$, $b'b''$ meet on the radical axis of these two circles. So again the points S' and S'' must lie on the same radical axis; *therefore* SS'S'', *the axis of similitude of the circles* C, C', C'', *is the radical axis of the circles* $aa'a''$, $bb'b''$.

(5). Since $a''b''$ passes through the centre of similitude of $aa'a''$, $bb'b''$, therefore (Art. 118) the tangents to these circles where it meets them intersect on the radical axis SS'S''. But this point of intersection must plainly be the pole of $a''b''$ with regard to the circle C''. Now since the pole of $a''b''$ lies on SS'S'', therefore (Art. 98) the pole of SS'S'' with regard to C'' lies on $a''b''$. Hence $a''b''$ is constructed by joining the radical centre to the pole of SS'S'' with regard to C''.

(6). Since the centre of similitude of two circles is on the line joining their centres, and the radical axis is perpendicular to that line, we learn (as in Art. 120) that the line joining the centres of $aa'a''$, $bb'b''$ passes through R, and is perpendicular to SS'S''.

CHAPTER IX.

PROPERTIES COMMON TO ALL CURVES OF THE SECOND DEGREE, DEDUCED FROM THE GENERAL EQUATION.

124. The most general form of the equation of the second degree is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, C, D, E, F are all constants.

The nature of the curve represented by this equation will vary according to the particular values of these constants. Thus we saw (Chap. V.), that in some cases this equation might represent two right lines, and (Chap. VI.), that for other values of the constants it might represent a circle. It is our object in this chapter to classify the different curves which can be represented by equations of the general form just written, and to obtain some of the properties which are common to them all.*

Five relations between the co-efficients are sufficient to determine a curve of the second degree. It is true that the general equation contains *six* constants, but it is plain that the nature of the curve does not depend on the *absolute magnitude* of these co-efficients, since, if we multiply or divide the equation by any constant, it will still represent the same curve. If, therefore, we write the equation in the form

$$\frac{A}{F}x^2 + \frac{B}{F}xy + \frac{C}{F}y^2 + \frac{D}{F}x + \frac{E}{F}y + 1 = 0,$$

it will be sufficient to determine the five quantities $\frac{A}{F}$, $\frac{B}{F}$, &c.

Thus, for example, a conic section can be described *through*

* We shall prove hereafter, that the section made by any plane in a cone standing on a circular base is a curve of the second degree, and, conversely, that there is no curve of the second degree which may not be considered as a *conic section*. It was in this point of view that these curves were first examined by geometers. We mention the property here, because we shall often find it convenient to use the terms "conic section" or "conic," instead of the longer appellation, "curve of the second degree."

five points. For each point $(x'y')$ through which the curve must pass, we have a relation between the co-efficients, viz.,

$$Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F = 0,$$

since the co-ordinates x', y' must satisfy the equation; and the five relations so obtained enable us to determine the five quantities, $\frac{A}{F}$, &c.

125. *Every right line must meet a curve of the second degree in two real, coincident, or imaginary points.*

Let us first consider the case of lines which pass through the origin. The truth of the proposition will then easily appear by transformation to polar co-ordinates. If the angle between the axes be ω, then for a line making angles α, β, with the axes, we saw (Art.15) that $x \sin \omega = \rho \sin \alpha$, $y \sin \omega = \rho \sin \beta$, or, as we shall write for shortness, $x = m\rho$, $y = n\rho$. Making these substitutions in the general equation, we have, to determine the length of the radius vector to either of the points where this line meets the curve, the quadratic,

$$(Am^2 + Bmn + Cn^2)\,\rho^2 + (Dm + En)\,\rho + F = 0.$$

We obtain the roots of this equation in a somewhat simpler form by solving for $\frac{1}{\rho}$ rather than for ρ, and we find

$$\frac{1}{\rho} = \frac{-(Dm + En) \pm \sqrt{\{(Dm + En)^2 - 4F\,.\,(Am^2 + Bmn + Cn^2)\}}}{2F}.$$

Since this equation always gives *two* values for ρ, we see, as in Art. 85, that every line through the origin will meet the curve in two real, coincident, or imaginary points.

The case of a line not passing through the origin is reduced to the former, by transferring the origin to any point on the line. Since transformation of co-ordinates cannot alter the degree of an equation (p. 48), the new equation must still be of the form

$$A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0;$$

and the co-ordinates of the points where any line $(my = nx)$ through the new origin meets the curve, are the *two* roots of a quadratic equation, precisely similar in form to that already given.

126. Although, in order to establish the theorem of the last Article, it was only necessary to show that the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

when transformed to parallel axes through any point $(x'y')$, was still of the same form, yet we shall often find it useful hereafter to know the actual values of the new co-efficients. We form the new equation by substituting $x + x'$ for x, and $y + y'$ for y (Art. 12), and we get

$$A(x+x')^2 + B(x+x')(y+y') + C(y+y')^2 + D(x+x') + E(y+y') + F = 0.$$

Arranging this equation according to the powers of the variables, we find that the co-efficients of x^2, xy, and y^2, will be, as before, A, B, C; that

the new D, $D' = 2Ax' + By' + D$;
the new E, $E' = 2Cy' + Bx' + E$;
the new F, $F' = Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F$.

Hence, *if the equation of a curve of the second degree be transformed to parallel axes through a new origin, the co-efficients of the highest powers of the variables will remain unchanged, while the new absolute term will be the result of substituting in the original equation the co-ordinates of the new origin.**

127. We meet with an apparent exception to the theorem of Art. 125, when the value of $m : n$ is such as to render

$$Am^2 + Bmn + Cn^2 = 0.$$

In this case the equation for determining ρ reduces to a simple equation, and it would seem as if the line $my = nx$ would only meet the curve in one point. However, on referring to the general value of $\frac{1}{\rho}$, given in Art. 125,

$$\frac{1}{\rho} = \frac{-(Dm + En) \pm \sqrt{\{(Dm + En)^2 - 4F.(Am^2 + Bmn + Cn^2)\}}}{2F},$$

we shall see that, although this line meets the curve in only one *finite* point, it will also meet it in another at an infinite distance. For, when $Am^2 + Bmn + Cn^2 = 0$, the quantity under the radical becomes equal to the rational part; therefore, when we give the radical the sign +, one value of $\frac{1}{\rho}$ becomes $= 0$, and, consequently, the corresponding value of ρ becomes infinite; while the other

* This is equally true for equations of any degree, as can be proved in like manner.

value of $\frac{1}{\rho}$, corresponding to the sign –, in general will remain finite, and will give $\rho = \frac{-F}{Dm + En}$.

Since two values of $m : n$ can in general be found, which will render $Am^2 + Bmn + Cn^2 = 0$, *there can be drawn through the origin two real, coincident, or imaginary lines, which will meet the curve at an infinite distance, and each of these lines will only meet the curve in* one *other point.*

We may prove, by the transformation of co-ordinates, as in Art. 125, that there are two directions in which lines can be drawn through *any* point to meet the curve at infinity; and, since it was proved, in Art. 126, that the co-efficients A, B, C were unaltered by transformation, we obtain for every point the very same quadratic, $Am^2 + Bmn + Cn^2 = 0$, to determine those directions. Hence, *if through any point two real lines can be drawn to meet the curve at infinity, parallel lines through any other point will meet the curve at infinity.**

If we multiply by ρ^2 the equation $Am^2 + Bmn + Cn^2 = 0$, and substitute for $m\rho$ and $n\rho$ their values x and y, we obtain for the equation of the two lines through the origin which meet the curve at infinity,

$$Ax^2 + Bxy + Cy^2 = 0.$$

128. The most important question we can ask, concerning the *form* of the curve represented by any equation, is, whether it be limited in every direction, or whether it extend in any direction to infinity. We have seen, in the case of the circle, that an equation of the second degree may represent a limited curve, while the case where it represents right lines shows us that it may also represent loci extending to infinity. It is necessary, therefore, to find a test whereby we may distinguish which class of locus is represented by any particular equation of the second degree.

With such a test we are at once furnished by the last Article. For if the curve be limited in every direction, *no* radius vector drawn from the origin to the curve can have an infinite value;

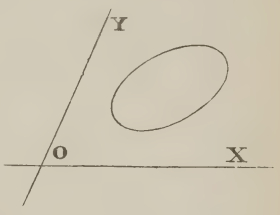
* This, indeed, is evident geometrically, since parallel lines may be considered as passing through the same point at infinity.

but we found in the last Article, that, in order that the radius vector should become infinite, we must have $Am^2 + Bmn + Cn^2 = 0$.

I. If now we suppose $B^2 - 4AC$ to be negative, the roots of this equation,

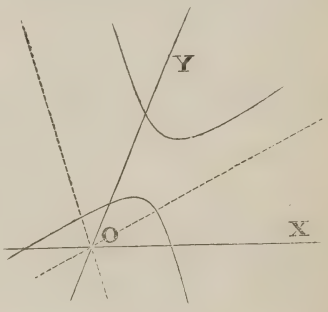
$$\frac{m}{n} = \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2C},$$

will be imaginary, and *no* real value of $m : n$ can be found which will render $Am^2 + Bmn + Cn^2 = 0$. In this case, therefore, no real line can be drawn to meet the curve at infinity, and *the curve will be limited in every direction.* We shall show, in the next chapter, that its form is that represented in the figure.

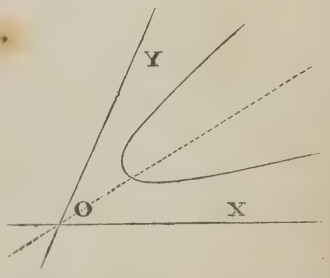


A curve of this class is called an *Ellipse.*

II. If $B^2 - 4AC$ be *positive*, the roots of the equation $Am^2 + Bmn + Cn^2 = 0$ will be real; consequently, there are two real values of $m : n$ which will render infinite the radius vector to one of the points where the line $(my = nx)$ meets the curve. Hence, in this case, two real lines $(Ax^2 + Bxy + Cy^2 = 0)$ can be drawn through the origin to meet the curve at infinity. A curve of this class is called an *Hyperbola*, and we shall show, in the next chapter, that its form is that represented in the figure.



III. If $B^2 - 4AC = 0$, the roots of the equation $Am^2 + Bmn + Cn^2 = 0$ will then be equal, and, therefore, the two directions in which a right line can be drawn to meet the curve at infinity, will in this case coincide. A curve of this class is called a *Parabola*, and we shall (Chap. XI.) show that its form is that here represented.



129. We shall here notice some of the particular forms which the general equation of the second degree may assume.

I. The circle is a particular form of the *ellipse*, for, since in the most general form of the equation of the circle $C = A$, $B = 2A\cos\omega$ (Art. 81), we have

$$B^2 - 4AC = -4A^2\sin^2\omega,$$

and, therefore, always negative.

II. If $B = 0$, while A and C have the same sign, the curve must be an ellipse; but if A and C have different signs, the curve is an hyperbola.

III. If either A or $C = 0$, and B not $= 0$, the quantity $B^2 - 4AC$ will reduce to B^2, which being essentially positive, the curve is an hyperbola.

In the case where $A = 0$ the axis of x is itself one of the lines which meet the curve at infinity, and where $C = 0$, the axis of y; these lines being in general given by the equation

$$Ax^2 + Bxy + Cy^2 = 0.$$

IV. If either A or C be $= 0$, and at the same time $B = 0$, then $B^2 - 4AC = 0$, and the curve is a parabola.

V. In general the curve will be a parabola, if the three first terms form a perfect square.

VI. If $F = 0$ the origin is on the curve (see note, p. 73).

VII. The curve will touch the axis of y if $E^2 = 4CF$, and the axis of x if $D^2 = 4AF$. This is proved precisely as at page 79.

We shall presently explain the effect of the suppositions D or $E = 0$. article 134

130. *From any point two tangents, real, coincident, or imaginary, can be drawn to a curve of the second degree.*

We found, in Art. 125, that the radii vectores to the points in which any line $my = nx$ meets the curve, are given by the equation

$$\frac{1}{\rho} = -\frac{(Dm+En) \pm \sqrt{\{(D^2-4AF)m^2 + 2(DE-2BF)mn + (E^2-4CF)n^2\}}}{2F}.$$

Now, if this line be a tangent, the two points in which it meets the curve must coincide, and, therefore, the values of ρ must be equal. This will be the case when the quantity under the radical $= 0$. The directions, therefore, in which tangents can be drawn from the origin are found from the quadratic

$$(D^2 - 4AF)\,m^2 + 2(DE - 2BF)\,mn + (E^2 - 4CF)\,n^2 = 0.$$

Therefore, as in Article 88, there are always *two*, and *only* two, such directions, real, coincident, or imaginary.

It is only necessary to notice particularly the case where these two directions coincide. If we apply the condition that the equation just obtained should have equal roots, we get

$$(D^2 - 4AF)\,(E^2 - 4CF) = (DE - 2BF)^2,$$

or
$$4F\,(AE^2 + CD^2 + FB^2 - BDE - 4ACF) = 0.$$

This will be satisfied, if $F = 0$, that is, if the origin be on the curve. Hence, *any point on the curve may be considered as the intersection of two coincident tangents*, just as we saw that any tangent might be considered as the line joining two coincident points.

The equation will also have equal roots if

$$AE^2 + CD^2 + FB^2 - BDE - 4ACF = 0.$$

Now we obtained this equation (p. 69) as the condition that the equation of the second degree should represent two right lines. To explain why we should here meet with this equation again, it must be remarked that by a tangent we mean in general a line which meets the curve in two coincident points; if then the curve reduce to two right lines, the only line which can meet the locus in two coincident points is the line drawn to the point of intersection of these right lines, and since *two* tangents can always be drawn to a curve of the second degree, both tangents must in this case coincide with the line to the point of intersection.

131. *The origin being on the curve, to find the equation of the tangent at it.*

We may find this equation by making $F = 0$ in the equation for determining the direction of the tangent given in the last Article, or else directly as follows:

Let the equation of the curve be

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = 0,$$

then the radii vectores to the points in which any line $my = nx$ meets it are found from the equation

$$(Am^2 + Bmn + Cn^2)\,\rho^2 + (Dm + En)\,\rho = 0.$$

Hence one value of ρ is always = 0, or the origin is always one point in which this line meets the curve, and the radius vector to the other is

$$\rho = \frac{-(Dm + En)}{Am^2 + Bmn + Cn^2}.$$

Now, if the line $my = nx$ be a tangent, the points in which it meets the curve will coincide, and, therefore, this value of ρ must also = 0. Hence $Dm + En = 0$, and, multiplying by ρ, and writing x and y for $m\rho$ and $n\rho$, the equation of the tangent is

$$Dx + Ey = 0.$$

132. *The origin not being on the curve, to find the equation of the line joining the points of contact of tangents through it.*

Let $m' : n'$ be one of the roots of the quadratic in Art. 130; then, since this value will render the radical = 0, we find, for the radius vector to the point where the tangent $m'y = n'x$ meets the curve,

$$\frac{1}{\rho} = -\frac{Dm' + En'}{2F}, \quad \text{or,} \quad Dm'\rho + En'\rho + 2F = 0;$$

the co-ordinates then of the points of contact must satisfy the equation

$$Dx + Ey + 2F = 0.$$

The point of contact of the tangent $m'y = n'x$ must, therefore, lie on the right line represented by this equation; so likewise must the point of contact of the other tangent, $m''y = n''x$; hence the line joining the points of contact of tangents through the origin is the line represented by the equation

$$Dx + Ey + 2F = 0.$$

This is the equation of a real line, whether the roots $m' : n'$, $m'' : n''$ be real or imaginary. We shall call this line, as in the case of the circle, the *polar* of the origin, and, conversely, we shall call the origin the *pole* of this line.

If the origin be on the curve, F = 0, and the equation of the polar becomes

$$Dx + Ey = 0;$$

we infer, therefore, from the last Article, that *the polar of any point on the curve is the tangent at the point.*

133. Having examined the case where the value of $m : n$ is such as to make the quantity under the radical = 0 in the value of

$\frac{1}{\rho}$ (Art. 125), we shall now consider the case where $m:n$ is such as to make the quantity *outside* the radical = 0, that is, when

$$Dm + En = 0, \text{ or } \frac{m}{n} = -\frac{D}{E}.$$

In this case the values of ρ will be of the form

$$\rho = \pm \sqrt{R}$$

(denoting the quantity under the radical by R); the values of ρ will, therefore, be equal with opposite signs. The points answering to these values are equidistant from the origin, and on opposite sides of it; therefore, the chord represented by the equation $Dx + Ey = 0$ is bisected at the origin.

This line, $Dx + Ey = 0$, is evidently parallel to the polar of the origin whose equation we saw was

$$Dx + Ey + 2F = 0.$$

Hence, *through any given point can in general be drawn one chord, which will be bisected at that point, and this chord will be parallel to the polar of the given point.*

134. There is one case, however, where more chords than one can be drawn, so as to be bisected, through a given point.

If, in the general equation, we had D = 0, E = 0, then the quantity $Dm + En$ would be = 0, whatever were the value of $m:n$; and we see, as in the last Article, that in this case *every* chord drawn through the origin would be bisected. The origin would then be called the *centre* of the curve. Now, although for any origin, taken arbitrarily, the quantities D and E are not = 0, yet we see, that if the curve have a centre, by taking this point for our origin, the quantities D and E will vanish, or, conversely, that if the axes be transformed to any new origin, so that the coefficients of x and y may vanish, then will the new origin be a centre of the curve.

In order to determine whether it be possible, by transformation of co-ordinates, to make the new D and E = 0, we have only to refer to the formulæ given in Art. 126, whence we find, that the co-ordinates of the new origin must fulfil the conditions

$$2Ax' + By' + D = 0,$$
$$2Cy' + Bx' + E = 0.$$

These *two* equations are sufficient to determine x' and y', and, *being linear*, can be satisfied by only *one* value of x and y; hence,

Curves of the second degree have in general one, *and* only *one centre.*

Its co-ordinates are found, by solving the above equations, to be

$$x = -\frac{BE - 2CD}{B^2 - 4AC}, \quad y = -\frac{BD - 2AE}{B^2 - 4AC}.$$

In the ellipse and hyperbola $B^2 - 4AC$ is always finite (Art. 128); but in the parabola $B^2 - 4AC = 0$, and the co-ordinates of the centre become infinite. The ellipse and hyperbola are hence often classed together as *central* curves, while the parabola is called a *non-central* curve. The student must be careful, however, to remember that, strictly speaking, *every* curve of the second degree has a centre, although in the case of the parabola this centre is situated at an infinite distance.

135. *To find the locus of the middle points of chords, parallel to a given line, of a curve of the second degree.*

We saw (Art. 133), that a chord through the origin $my = nx$ is bisected if $Dm + En = 0$. Now, transforming the origin to any point, it appears, in like manner, that a parallel chord will be bisected at the new origin if m times the new $D + n$ times the new $E = 0$, or (Art. 126),

$$m(2Ax' + By' + D) + n(Bx' + 2Cy' + E) = 0.$$

This, therefore, is a relation which must be satisfied by the co-ordinates of the new origin, if it be the middle point of a chord parallel to $my = nx$. Hence, the middle point of any parallel chord must lie on the right line

$$m(2Ax + By + D) + n(Bx + 2Cy + E) = 0,$$

which is, therefore, the required locus.

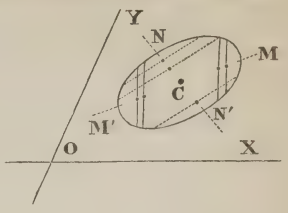
Every right line bisecting a system of parallel chords is called a *diameter*, and the lines which it bisects are called its *ordinates.*

The form of the equation shows (Art. 48), that every diameter must pass through the intersection of the two lines

$$2Ax + By + D = 0, \text{ and } 2Cy + Bx + E = 0;$$

and, on turning to the equations from which we determined the

co-ordinates of the centre (Art. 134), we infer, that *every diameter passes through the centre of the curve.*



Since

$$m(2Ax+By+D)+n(Bx+2Cy+E)=0$$

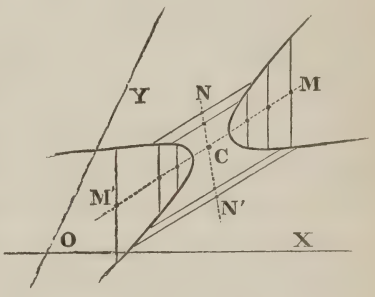
is the equation of the diameter bisecting chords parallel to $my = nx$, it appears, by making m and n alternately = 0, that

$$2Ax + By + D = 0$$

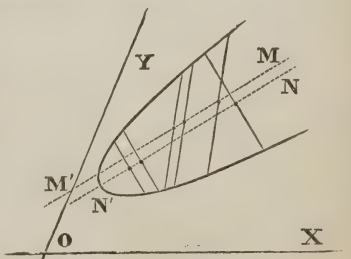
is the equation of the diameter bisecting chords parallel to the axis of x, and that

$$2Cy + Bx + E = 0$$

is the equation of the diameter bisecting chords parallel to the axis of y.



In the parabola $B^2 = 4AC$, or $\frac{2A}{B} = \frac{B}{2C}$, and hence the line $2Ax + By + D = 0$ is parallel to the line $2Cy + Bx + E = 0$; consequently, *all diameters of a parabola are parallel to each other.* This, indeed, is evident, since we have proved that all diameters of any conic section must pass through the centre, which, in the case of the parabola, is at an infinite distance; and since parallel right lines may be considered as meeting in a point at infinity.*



The familiar example of the circle will sufficiently illustrate to the beginner the nature of the diameters of curves of the second

* Hence, given any conic section, we can find its centre geometrically. For if we draw any two parallel chords, and join their middle points, we have one diameter. In like manner we can find another diameter. Then, if these two diameters be parallel, the curve is a parabola, but if not, the point of intersection is the centre.

degree. He must observe, however, that diameters do not in general, as in the case of the circle, cut their ordinates at right angles. In the parabola, for instance, the direction of the diameter being invariable, while that of the ordinates may be any whatever, the angle between them *may take any possible value.*

136. The direction of the diameters of a parabola is the same as that of the line through the origin which meets the curve at an infinite distance.

For the lines through the origin which meet the curve at infinity are (Article 127)

$$Ax^2 + Bxy + Cy^2 = 0,$$

or, writing for B its value, $\sqrt{(4AC)}$,

$$(\sqrt{A}x + \sqrt{C}y)^2 = 0.$$

But the diameters are parallel to $2Ax + By = 0$, by the last Article, which, if we write for B the same value, $\sqrt{(4AC)}$, will also reduce to

$$\sqrt{A}x + \sqrt{C}y = 0.$$

Hence every diameter of the parabola meets the curve once at infinity, and, therefore, can only meet it in one finite point.

137. *If two diameters of a conic section be such, that one of them bisects all chords parallel to the other, then, conversely, the second will bisect all chords parallel to the first.*

The equation of the diameter which bisects chords parallel to $my = nx$ is (Art. 135)

$$(2Am + Bn)x + (Bm + 2Cn)y + Dm + En = 0.$$

If then this be parallel to $m'y = n'x$, we must have

$$\frac{m'}{n'} = -\frac{Bm + 2Cn}{2Am + Bn},$$

or

$$2Amm' + B(m'n + mn') + 2Cnn' = 0.$$

But the symmetry of the equation shows that it is also the condition that the line $my = nx$ should be parallel to the diameter bisecting the chord $m'y = n'x$.

Diameters so related, that each bisects every chord parallel to the other, are called *conjugate diameters.**

* It is evident that none but central curves can have conjugate diameters, since in the parabola the direction of all diameters is the same.

If in the general equation $B = 0$, the axes will be parallel to a pair of conjugate diameters.

For the diameter bisecting chords parallel to the axis of x will, in this case, become $2Ax + D = 0$, and will, therefore, be parallel to the axis of y.

In like manner, the diameter bisecting chords parallel to the axis of y will, in this case, be $2Cy + E = 0$, and will, therefore, be parallel to the axis of x.

138. *To find the equation of the tangent at any point* $(x'y')$.

The process by which (p. 76) we found the equation of the tangent to a circle expressed by the general equation is applicable to all curves of the second degree, if we make the coefficient of y^2, C instead of A. We find thus for the equation of the chord,

$$\frac{y - y'}{x - x'} = -\frac{A(x' + x'') + By'' + D}{C(y' + y'') + Bx' + E};$$

and for the equation of the tangent,

$$\frac{y - y'}{x - x'} = -\frac{2Ax' + By' + D}{2Cy' + Bx' + E}.$$

Or reducing, and remembering that $x'y'$ satisfies the equation of the curve,

$$(2Ax' + By' + D)x + (2Cy' + Bx' + E)y + Dx' + Ey' + 2F = 0.$$

139. *To find the equation of the polar of any point* $x'y'$.

The method used in Art. 90 is applicable, word for word, to the general equation of the second degree, and shows that the equation of the polar is exactly similar in form to the equation of the tangent, with the only difference that the point $x'y'$ is no longer on the curve. Or the same result may be obtained by transformation of co-ordinates as follows:

We saw (Art. 132), that the equation of the polar of the origin was $Dx + Ey + 2F = 0.$

Now, if we transform the axes to any new point $(x'y')$, the polar of this new origin must be $D'x + E'y + 2F' = 0$; or (Art. 126),

$$(2Ax' + By' + D)x + (Bx' + 2Cy' + E)y + 2(Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F) = 0.$$

This equation is transformed back to the old origin by writing $x - x'$ for x, and $y - y'$ for y, and is, therefore,

$$(2Ax' + By' + D)(x - x') + (Bx' + 2Cy' + E)(y - y') + 2(Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F = 0;$$

whence, by reduction, we find the equation of the polar,

$$(2Ax' + By' + D)x + (Bx' + 2Cy' + E)y + Dx' + Ey' + 2F = 0.$$

140. It is evident that a chord drawn through any point, so as to be bisected at it, must be an ordinate to the diameter passing through that point. Hence the theorem of Art. 133 admits of being stated as follows:

The polar of any point is parallel to the ordinates of the diameter passing through that point. This includes, as a particular case: *The tangent at the extremity of any diameter is parallel to the ordinates of that diameter.* Or, again, in the case of central curves, since the ordinates of any diameter are parallel to the conjugate diameter, we infer that, *The polar of any point on a diameter of a central curve is parallel to the conjugate diameter.*

141. *If any point* $(x''y'')$ *be taken on the polar of* $(x'y')$, *its polar must pass through* $(x'y')$.

For, the condition that $(x''y'')$ should lie on the polar of $(x'y')$ is (Art. 139),

$$(2Ax' + By' + D)x'' + (2Cy' + Bx' + E)y'' + Dx' + Ey' + 2F = 0.$$

But this may be arranged

$$(2Ax'' + By'' + D)x' + (2Cy'' + Bx'' + E)y' + Dx'' + Ey'' + 2F = 0,$$

and is, therefore, also the condition that $(x'y')$ should lie on the polar of $(x''y'')$.

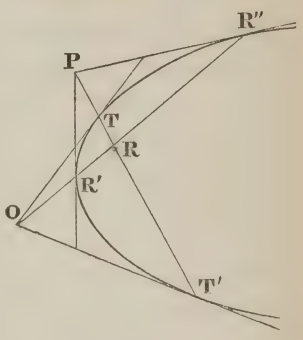
Since, if a third point $(x'''y''')$ be taken on the polar of $(x'y')$ its polar must also pass through $(x'y')$, the preceding proposition may be stated thus:

The intersection of any two lines is the pole of the line joining their poles; and, conversely: *The line joining any two points is the polar of the intersection of the polars of these points.* For here $(x''y'')$ and $(x'''y''')$ are the two points, and $(x'y')$, which is the pole of the line joining those points, is also the intersection of their polars.

Generally, *if any point move along a fixed right line, its polar must always pass through a fixed point, namely, the pole of the fixed line.*

This is an easy consequence from the other theorems established in this Article; or it may be deduced independently from the equation of the polar, by the help of the principle laid down in Art. 71.

142. *If any radius vector drawn through the origin* O *meet the curve in* R′ *and* R″, *and if* OR *be taken an harmonic mean between* OR′ *and* OR″: *to prove that* R *lies on the polar of* O.



We found (Art. 125) that OR′, OR″ were determined by the quadratic

$$(Am^2 + Bmn + Cn^2)\,\rho^2 + (Dm + En)\,\rho + F = 0.$$

Hence, by the theory of equations,

$$\frac{1}{OR'} + \frac{1}{OR''} = -\frac{Dm + En}{F},$$

and, therefore,

$$\frac{2}{OR} = -\frac{Dm + En}{F}.$$

In order to find the locus of R we must write x and y for $m.OR$ and $n.OR$, and the equation of the locus is

$$Dx + Ey + 2F = 0,$$

that is, the equation of the polar of the origin.

Hence, *any line drawn through a point is cut harmonically by the point, the curve, and the polar of the point.**

143. *If two lines be drawn through any point, and the points joined where they meet a curve of the second degree, the joining lines will intersect on the polar of that point.*

The reader, on referring to the proof given (p. 95) of this property in the case of the circle, will find that the same proof will apply, word for word, to the general case of conic sections, since no use was there made of the equality of the co-efficients of x^2 and y^2.

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* For an enumeration of some of the particular cases included in this theorem, the reader is referred to the section on the harmonic and anharmonic properties of conic sections, Chap. XIV.

If through a point O *any line* OR′ *be drawn, the tangents at* R′ *and* R″ *will meet on the polar of* O.

This is a particular case of the preceding theorem, namely, where the two lines are supposed to coincide; or else it follows immediately from Art. 141, since the pole of any chord through O must lie on TT′; and by the pole of the line we mean the intersection of tangents at the points where it meets the curve.*

144. *If any line* (OR) *be drawn through a point* (O), *and* (P) *the pole of that line, be joined to* O, *then the four lines,* OP, OT, OR, OT′, *will form an harmonic pencil* (OT *and* OT′ *being the tangents from* O).

For, since OR is the polar of P, by Art. 142 PTRT′ is cut harmonically, therefore, OP, OT, OR, OT′, form an harmonic pencil.

145. The theorem of Art. 142 may also be proved by a process precisely similar to that employed at page 82. We may seek the ratio in which the line joining two points is cut by the curve. Substituting $\frac{lx'' + mx'}{l+m}$, $\frac{ly'' + my'}{l+m}$, for x and y, we have to determine $l : m$ the quadratic

$$l^2 (Ax''^2 + Bx''y'' + Cy''^2 + Dx'' + Ey'' + F)$$
$$+ lm\{(2Ax'' + By'' + D)\, x' + (Bx'' + 2Cy'' + E)\, y' + Dx'' + Ey'' + 2F\}$$
$$+ m^2 (Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F) = 0.$$

Now if $x''y''$ be on the polar of $x'y'$ the coefficient of lm vanishes, the roots of the equation are of the form $l = \pm \mu m$, and the line joining the points is cut harmonically.

The same equation enables us also to write down the equation of the pair of tangents drawn from any point to the curve. For if $x''y''$ lie on either of the tangents through $x'y'$, the equation for $l : m$ must have equal roots, and $x''y''$ must therefore satisfy the equation

$$4(Ax^2 + Bxy + Cy^2 + Dx + Ey + F)(Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F)$$
$$= \{(2Ax' + By' + D)\, x + (Bx' + 2Cy' + E)\, y + Dx' + Ey' + 2F\}^2.$$

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* From this property the polar of a point might have been *defined* as the locus of the intersection of tangents at the extremities of any chord passing through the point. This definition applies, whether the point be within or without the conic.

145. *If through any point* O *two chords be drawn, meeting the curves in the points* R′, R″, S′, S″, *then the ratio of the rectangles* $\frac{OR'.OR''}{OS'.OS''}$ *will be constant, whatever be the position of the point* O, *provided that the* direction *of the lines* OR, OS *be constant.*

For, from the equation given to determine ρ in Art. 125, it appears that

$$OR'.OR'' = \frac{F}{Am^2 + Bmn + Cn^2}.$$

In like manner

$$OS'.OS'' = \frac{F}{Am'^2 + Bm'n' + Cn'^2};$$

hence

$$\frac{OR'.OR''}{OS'.OS''} = \frac{Am'^2 + Bm'n' + Cn'^2}{Am^2 + Bmn + Cn^2}.$$

Now we saw (Art. 126), that if we transformed the axes to any new origin O′, the quantities A, B, C would remain unaltered, and (Art. 125) that the quantities m, n, m', n' depend only on the angles which the radius vector makes with the axes, and are therefore constant while the direction of this radius vector is constant; the ratio just given will therefore remain constant.

The theorem of this Article may be otherwise stated thus: *If through two fixed points* O *and* O′ *any two parallel lines* OR *and* O′ρ *be drawn, then the ratio of the rectangles* $\frac{OR'.OR''}{O'\rho'.O'\rho''}$ *will be constant, whatever be the direction of these lines.*

For, the first rectangle is, as we saw,

$$\frac{F}{Am^2 + Bmn + Cn^2},$$

and the second rectangle is

$$\frac{F'}{Am^2 + Bmn + Cn^2},$$

(F′ being the new absolute term when the equation is transferred to O′ as origin), the ratio of these rectangles $= \frac{F}{F'}$, and is, therefore, independent of m and n.

This theorem is the generalization of Euclid, III. 35, 36.

147. The theorem of the last Article includes under it several particular cases, which it is useful to notice separately.

I. Let O′ be the centre of the curve, then $O'\rho' = O'\rho''$ and the quantity $O'\rho' . O'\rho''$ becomes the square of the semidiameter parallel to OR′. Hence, *The rectangles under the segments of two chords which intersect are to each other as the squares of the diameters parallel to those chords.*

II. Let the line OR be a tangent, then OR′ = OR″, and the quantity OR′. OR″ becomes the square of the tangent; and, since two tangents can be drawn through the point O, we may extract the square roots of the ratio found in the last paragraph, and infer that *Two tangents drawn through any point are to each other as the diameters to which they are parallel.*

III. Let the line OO′ be a diameter, and OR, O′ρ, parallel to its ordinates, then OR′ = OR″ and $O'\rho' = O'\rho''$. Let the diameter meet the curve in the points A, B, then $\frac{OR^2}{AO.OB} = \frac{O'\rho^2}{AO'.O'B}$. Hence, *The squares of the ordinates of any diameter are proportional to the rectangles under the segments which they make on the diameter.*

148. There is one case in which the theorem of Article 146 becomes no longer applicable, namely, when the line OS is parallel to one of the lines which meet the curve at infinity; the segment OS″ is then infinite, and OS only meets the curve in one finite point. We propose, in the present Article, to inquire whether, in this case, the ratio $\frac{OS'}{OR' . OR''}$ will be constant.

Let us, for simplicity, take the line OS for our axis of x, and OR for the axis of y. Since the axis of x is parallel to one of the lines which meet the curve at infinity, the term A will = 0 (Art. 129, III.), and the equation of the curve will be of the form

$$Bxy + Cy^2 + Dx + Ey + F = 0.$$

Making $y = 0$, the intercept on the axis of x is found to be $OS' = -\frac{F}{D}$; and making $x = 0$, the rectangle under the intercepts on the axis of y is $= \frac{F}{C}$.

Hence $$\frac{OS'}{OR' \cdot OR''} = -\frac{C}{D}.$$

Now, if we transform the axes to any parallel axes (Art. 126), C will remain unaltered, and the new $D = Bx' + D$.

Hence the new ratio will be

$$-\frac{C}{Bx' + D}.$$

Now, if the curve be a parabola, $B = 0$, and this ratio is constant; hence, *if a line parallel to a given one meet any diameter* (Art. 136) *of a parabola, the rectangle under its segments is in a constant ratio to the intercept on the diameter.*

If the curve be a hyperbola the ratio will only be constant while x' is constant; hence, *if a chord of a hyberbola be met by two lines parallel to an asymptote, the lengths of these lines are proportional to the rectangles under the segments into which each divides the chord.*

CHAPTER X.

EQUATIONS OF THE SECOND DEGREE REFERRED TO THE CENTRE AS ORIGIN.

149. In investigating the properties of the ellipse and hyperbola, we shall find our equations much simplified by choosing the centre for the origin of co-ordinates. If we transform the general equation of the second degree to the centre as origin, we saw (Art. 134) that the co-efficients of x and y will = 0 in the transformed equation, which will be of the form

$$Ax^2 + Bxy + Cy^2 + F' = 0.$$

It is sometimes useful to know the value of F′ in terms of the coefficients of the first given equation. We saw (Art. 126) that

$$F' = Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F,$$

where x', y', are the co-ordinates of the centre, or

$$x' = \frac{2CD - BE}{B^2 - 4AC}, \text{ and } y' = \frac{2AE - BD}{B^2 - 4AC}.$$

The calculation of this may be facilitated by putting F′ into the form

$$F' = \tfrac{1}{2}\{(2Ax' + By' + D)x' + (2Cy' + Bx' + E)y' + Dx' + Ey' + 2F\}.$$

The first two terms must be rendered = 0 by the co-ordinates of the centre, and the last

$$= D \cdot \frac{2CD - BE}{B^2 - 4AC} + E \cdot \frac{2AE - BD}{B^2 - 4AC} + 2F.$$

Hence

$$F' = \frac{AE^2 + CD^2 + FB^2 - BDE - 4ACF}{B^2 - 4AC}.$$

150. If the numerator of this fraction were = 0, the transformed equation would be reduced to the form

$$Ax^2 + Bxy + Cy^2 = 0,$$

and would, therefore (Art. 73), represent two real or imaginary right lines, according as $B^2 - 4AC$ is positive or negative. Hence, as we have already seen, p. 69, the condition that the general equation of the second degree should represent two right lines, is

$$AE^2 + CD^2 + FB^2 - BDE - 4ACF = 0.$$

For it must plainly be fulfilled, in order that when we transfer the origin to the point of intersection of the right lines, the absolute term may vanish.

151. To return to the general case. We saw (Art. 137) that if the axes be a pair of *conjugate diameters*, the quantity B must = 0; hence the equation must be of the form

$$Ax^2 + Cy^2 = F.$$

It is evident, now, that any line parallel to either axis is bisected by the other, for if we give to x any value, we obtain equal and opposite values for y. Now the angle between conjugate diameters is not in general right, as it is in the case of the circle; but we shall show that there is always *one pair* of conjugate diameters which cut each other at right angles. These diameters are called the *axes* of the curve, and the points where they meet the curve are called its *vertices*.

The equation of the diameter conjugate to $my = nx$ is (Art. 137), in general,

$$m(2Ax + By + D) + n(2Cy + Bx + E) = 0,$$

and the condition that this should be perpendicular to $my = nx$ is (Art. 38)

$$(2Am + Bn)\,n - (Bm + 2Cn)\,m = 0,$$

or

$$Bm^2 - 2\,(A - C)\,mn - Bn^2 = 0\ ;$$

or, multiplying by ρ^2, and writing x, y for $m\rho, n\rho$,

$$Bx^2 - 2\,(A - C)\,xy - By^2 = 0.$$

This is the equation of two real lines at right angles to each other (Art. 74); we perceive, therefore, that central curves have two, and only two, conjugate diameters at right angles to each other.

152. On referring to Art. 75 it will be found, that the equation which we have just obtained for the *axes* of the curve is the same as that of the lines bisecting the internal and external angles between the real or imaginary lines represented by the equation

$$Ax^2 + Bxy + Cy^2 = 0.$$

It was proved (Art. 127), that this equation represented the real or imaginary lines drawn through the origin to meet the curve at infinity, and also that the direction of these lines remains the same, whatever be the point through which they are drawn. Again, it was proved (Art. 127) that each of these lines will, in general, meet the curve in one other point, at a distance from the origin,

$$\rho = \frac{-F}{Dm + En}.$$

But if the point through which the lines are drawn be the centre, we have $D = 0$, $E = 0$, and this distance will *also* become infinite. Hence two lines can be drawn through the centre, which will meet the curve in *two coincident* points at infinity, and, consequently, in no other point, real or imaginary. These lines are imaginary in the case of the ellipse, but in the hyperbola they are real, and are called the *asymptotes* of the curve.

We shall show hereafter that though the asymptotes do not meet the curve at any finite distance, yet that the further they are produced, the more nearly they approach the curve.

Since we have showed that the asymptotes meet the curve in two coincident points at infinity, it appears, from our definition of a tangent, that *the asymptotes may be considered as tangents to the curve whose points of contact are at an infinite distance.*

Hence, since the asymptotes pass through the centre, two real or imaginary tangents may be drawn through the centre, each of whose points of contact will be at an infinite distance, and, therefore, the line joining those points of contact will be altogether at an infinite distance. Hence, from our definition of poles and polars (Art. 132), *the centre may be considered as the pole of a line situated altogether at an infinite distance.**

From the present Article we see that the *axes* of the curve are the diameters which bisect the angles between the asymptotes, and (note, p. 67) that the axes will be real whether the asymptotes be real or imaginary, that is to say, whether the curve be an ellipse or an hyperbola.

153.† We proved that if we transform the equation of the second degree from any given co-ordinate axes to the *axes* of the curve, it must take the form

$$Ax^2 + Cy^2 = F.$$

We might have used this principle to discover the axes of the curve. Let the given axes be rectangular; then, if we turn them round through any angle θ, we have (Art. 14) to substitute for x, $x\cos\theta - y\sin\theta$, and for y, $x\sin\theta + y\cos\theta$; the equation will, therefore, become

$$A(x\cos\theta - y\sin\theta)^2 + B(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + C(x\sin\theta + y\cos\theta)^2 = F;$$

or, arranging the terms, we shall have

$$\text{the new } A = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta;$$
$$\text{the new } B = 2C\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) - 2A\sin\theta\cos\theta;$$
$$\text{the new } C = A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta.$$

Now, if we put the new $B = 0$, we get the very same equation to determine $\tan\theta$, which we had, in Art. 151, to determine m: viz. This affords an independent proof, that the equation referred to the *axes* will want the term xy. This equation also gives us a

* This inference may be confirmed from the equation of the polar of the origin, $Dx + Ey + 2F = 0$, which, if the centre be the origin (D and E being then $= 0$), reduces to $F = 0$ an equation which, we saw (Art. 72), represents a line situated altogether at infinity.

† This Article and the next may, on first reading, be omitted.

simple expression for the angle made by the new axes with the old, namely,

$$\tan 2\theta = \frac{B}{A - C}.$$

154. Our chief object in giving the formulæ of transformation here was in order to demonstrate the following theorem. We know that the nature of the curve depends on the sign of the quantity $B^2 - 4AC$. Now, as the nature of the curve cannot be altered by any transformation of co-ordinates, it is important to show that neither can the sign of the quantity $B^2 - 4AC$.

First. If the axes be transformed to any parallel axes this is evident, since (Art. 126) the co-efficients A, B, C, remain unchanged.

Secondly. If the axes be transformed from one rectangular system to another, although A, B, C will be changed, yet the quantity $B^2 - 4AC$ will remain unaltered, for the values in the last Article may be written

$$2A' = A + C + B \sin 2\theta + (A - C) \cos 2\theta,$$
$$2C' = A + C - B \sin 2\theta - (A - C) \cos 2\theta.$$

Hence

$$4A'C' = (A + C)^2 - \{B \sin 2\theta + (A - C) \cos 2\theta\}^2.$$

But

$$B'^2 = \{B \cos 2\theta - (A - C) \sin 2\theta\}^2;$$

therefore,

$$B'^2 - 4A'C' = B^2 + (A - C)^2 - (A + C)^2 = B^2 - 4AC.$$

Thirdly. If the axes be transformed from a rectangular system to an oblique system containing an angle $= \omega$.

Since $B^2 - 4AC$ is the same, whatever rectangular axes we use, we may suppose the axis of x not to be changed. Then (Art. 13) we must substitute for x, $x + y \cos \omega$, and for y, $y \sin \omega$. Hence we shall have

$$A' = A;$$
$$B' = 2A \cos \omega + B \sin \omega;$$
$$C' = A \cos^2 \omega + B \sin \omega \cos \omega + C \sin^2 \omega.$$

Therefore, $B'^2 - 4A'C' = (B^2 - 4AC) \sin^2 \omega$.

Its sign must, consequently, be the same as that of $B^2 - 4AC$.

THE EQUATION REFERRED TO THE AXES.

155. We saw that the equation referred to the axes was of the form

$$Ax^2 + Cy^2 = F,$$

C being positive in the case of the ellipse, and negative in that of the hyperbola (Art. 133, II.)

The equation of the ellipse may be written in the following more convenient form :

Let the intercept made by the ellipse on the axis of x be $= a$, then a is found by making $y = 0$ and $x = a$ in the equation of the curve, or $Aa^2 = F$, and $a^2 = \frac{F}{A}$. In like manner, if b be the intercept on the axis of y, $b^2 = \frac{F}{C}$. Hence the equation of the ellipse,

$$\frac{A}{F}x^2 + \frac{C}{F}y^2 = 1,$$

may be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The equation of the hyperbola, which, we saw, only differs from that of the ellipse in the sign of the co-efficient of y^2, may be written in the corresponding form,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

156. We now proceed from these equations to investigate the figure of the curves, and commence with the equation of the ellipse.

Since we may choose whichever axis we please for the axis of x, we shall suppose that we have chosen the axes so that a may be greater than b.

The length of any semidiameter of the ellipse can be found, by writing $\rho \cos\theta$ for x, and $\rho \sin\theta$ for y, in the preceding equation. We thus get

$$\frac{1}{\rho^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2},$$

or

$$\rho^2 = \frac{a^2 b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta}$$

This equation may otherwise be written in either of the forms

$$\rho^2 = \frac{a^2 b^2}{a^2 - (a^2 - b^2)\cos^2\theta} = \frac{a^2 b^2}{b^2 + (a^2 - b^2)\sin^2\theta}.$$

It is customary to use the following abbreviations,

$$a^2 - b^2 = c^2; \quad \frac{a^2 - b^2}{a^2} = e^2;$$

and the quantity e is called the *eccentricity* of the curve.

Dividing by a^2, the numerator and denominator of the fraction just found, we obtain the form most commonly used of the *polar equation of the ellipse*, the *centre* being the pole,

$$\rho^2 = \frac{b^2}{1 - e^2\cos^2\theta}.$$

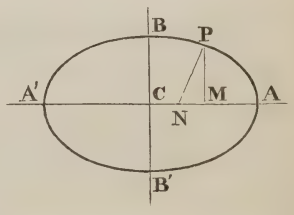
Now, the *least* value that $b^2 + (a^2 - b^2)\sin^2\theta$ can have, is when $\theta = 0$; therefore, since

$$\rho^2 = \frac{a^2 b^2}{b^2 + (a^2 - b^2)\sin^2\theta},$$

the *greatest* value of ρ is the intercept on the axis of x, and is $= a$.

Again, the *greatest* value of $b^2 + (a^2 - b^2)\sin^2\theta$, is, when $\sin\theta = 1$, or $\theta = 90°$; hence the *least* value of ρ is the intercept on the axis of y, and is $= b$. The greatest line, therefore, that can be drawn through the centre is the axis of x, and the least line, the axis of y.

From this property these lines are called the axis *major* and the axis *minor* of the curve. It is plain that the smaller θ is, the greater ρ will be; hence, *the nearer any diameter is to the axis major, the greater it will be*. The form of the curve will, therefore, be that here represented.



We obtain the same value of ρ whether we suppose $\theta = \alpha$, or $\theta = -\alpha$. Hence, *Two diameters which make equal angles with the axis will be equal.* And it is easy to show that the converse of this theorem is also true.

This property enables us, being given the centre of a conic section, to determine its axes geometrically. For if, with any semidiameter as radius, we describe a concentric circle, and draw a semidiameter to the point where this circle cuts the conic section

again, these semidiameters will be equal, being radii of the same circle; and by the theorem just proved, the axes of the conic will be the lines internally and externally bisecting the angle between them.

157. The equation of the ellipse can be put into another form, which will make the figure of the curve still more apparent. If we solve for y we get

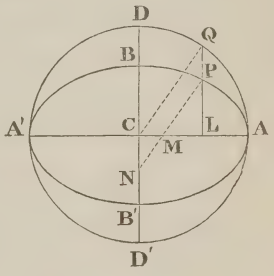
$$y = \frac{b}{a}\sqrt{(a^2 - x^2)}.$$

Now, if we describe a concentric circle with the radius a, its equation will be

$$y = \sqrt{(a^2 - x^2)}.$$

Hence we derive the following construction:

"*Describe a circle on the axis major, and take on each ordinate* LQ *a point* P, *such that* LP *may be to* LQ *in the constant ratio* $\frac{b}{a}$, *then the locus of* P *will be the required ellipse.*"



Hence the circle described on the axis major lies wholly *without* the curve. We might, in like manner, construct the ellipse, by describing a circle on the axis minor, and *increasing* each ordinate in the constant ratio $\frac{a}{b}$.

Hence the circle described on the axis minor lies wholly *within* the curve.

We see also, that the equation of the circle is the particular form which the equation of the ellipse assumes when we suppose $b = a$.

158. We now proceed to investigate the figure of the hyperbola from its equation,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The intercept on the axis of x is evidently $= \pm a$, but that on the axis of y, being found from the equation $\frac{y^2}{b^2} = -1$ is imaginary; the axis of y, therefore, does not meet the curve in real points.

Since we have chosen for our axis of x the axis which meets

the curve in real points, we are not entitled to suppose a greater than b. The terms, therefore, axis major and axis minor, become inapplicable, and we shall call the axis of x the *transverse* axis, and the axis of y the *conjugate* axis.

159. *To find the length of any semidiameter of the hyperbola.*

Transforming to polar co-ordinates, as in the case of the ellipse, we get

$$\rho^2 = \frac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta} = \frac{a^2b^2}{b^2 - (a^2+b^2)\sin^2\theta} = \frac{a^2b^2}{(a^2+b^2)\cos^2\theta - a^2}.$$

Since formulæ concerning the ellipse are altered to the corresponding formulæ for the hyperbola by changing the sign of b^2, we must, in this case, use the abbreviation c^2 for $a^2 + b^2$, and e^2 for $\frac{a^2+b^2}{a^2}$, the quantity e being called the *eccentricity* of the hyperbola.

Dividing then by a^2, the numerator and denominator of the last found fraction, the polar equation of the hyperbola is obtained,

$$\rho^2 = \frac{b^2}{e^2\cos^2\theta - 1},$$

a form only differing from the polar equation of the ellipse in the sign of b^2.

Now, the denominator $b^2 - (a^2 + b^2)\sin^2\theta$ will plainly be greatest when $\theta = 0$, therefore, in the same case, ρ will be least; or *the transverse axis is the shortest line which can be drawn from the centre to the curve.*

As θ increases ρ continually increases, until

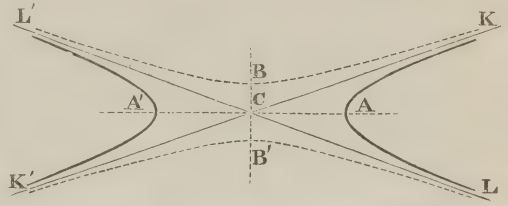
$$\sin\theta = \frac{b}{\sqrt{(a^2+b^2)}}, \quad \left(\text{or } \tan\theta = \frac{b}{a}\right),$$

when the denominator of the value of ρ becomes $= 0$, and ρ becomes infinite. After this value of θ, ρ^2 becomes negative, and the diameters cease to meet the curve in real points until

$$\sin\theta = -\frac{b}{\sqrt{(a^2+b^2)}}, \quad \left(\text{or } \tan\theta = -\frac{b}{a}\right),$$

when ρ again becomes infinite. It then decreases regularly as θ increases, until θ becomes $= 180°$, when it again receives its minimum value $= a$.

The form of the hyperbola, therefore, is that represented by the dark curve on the figure.



160. We found that the axis of y does not meet the hyperbola in real points, since we obtained the equation $y^2 = -b^2$ to determine its point of intersection with the curve. We shall, however, still mark off on the axis of y portions, CB, $CB' = \pm b$, and we shall find that the length CB has an important connexion with the curve, and may be conveniently called an axis of the curve. In like manner, if we obtained an equation to determine the length of any other diameter, of the form $\rho^2 = -R^2$, although this diameter cannot meet the curve, yet if we measure on it from the centre lengths $= \pm R$, these lines may be conveniently spoken of as diameters of the hyperbola.

The *locus* of the extremities of these diameters which do not meet the curve, is at once found by changing the sign of ρ^2 in the equation of the curve

$$\frac{1}{\rho^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}$$

to be

$$\frac{1}{\rho^2} = \frac{\sin^2\theta}{b^2} - \frac{\cos^2\theta}{a^2},$$

or

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

This is the equation of a hyperbola having the axis of y for its axis meeting it in real points, and the axis of x for the axis meeting it in imaginary points. It is represented by the dotted curve on the figure, and is called the hyperbola *conjugate* to the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

161. We proved (Art. 159) that the diameters answering to $\tan\theta = \pm\frac{b}{a}$ meet the curve at infinity; they are, therefore, the same as the lines called, in Art. 152, the *asymptotes* of the curve.

They are the lines CK, CL on the figure, and evidently separate those diameters which meet the curve in real points from those which meet it in imaginary points.

The expression $\tan\theta = \pm\frac{b}{a}$ enables us, being given the axes in magnitude and position, to find the asymptotes, for, if we form a rectangle by drawing parallels to the axes through B and A, then the asymptote CK must be the diagonal of this rectangle.

Again,

$$\cos\theta = \frac{a}{\sqrt{(a^2+b^2)}} = \frac{1}{e}.$$

But, since the asymptotes make equal angles with the axis of x, the angle which they make with each other must be $= 2\theta$. Hence, *being given the eccentricity of a hyperbola, we are given the angle between the asymptotes*, which is double the angle whose secant is the eccentricity.

We saw (Art. 152) that the equations of the asymptotes were found by putting the highest terms of the equation = 0, the centre being the origin. Therefore, the equations of the asymptotes to the curve

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are contained in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

Thus we see that two conjugate hyperbolæ have the same asymptotes.

The lines drawn through any point to meet the curve at infinity must (Art. 127) be parallel to the lines drawn through the centre to meet the curve at infinity, that is (Art. 152), must be parallel to the asymptotes. Hence (Art. 129), the axis of x will be parallel to an asymptote if the co-efficient of $x^2 = 0$, and the axis of y will be parallel to an asymptote if the coefficient of $y^2 = 0$. If the axes be the asymptotes themselves, we must have, in addition, the co-efficients of x and $y = 0$ (since the asymptotes intersect at the centre); and the equation of the hyperbola, referred to the asymptotes as axes, takes the simple form,

$$xy = \mathrm{F}.$$

THE TANGENT.

162. We now proceed to investigate some of the properties of the ellipse and hyperbola. We shall find it convenient to consider both curves together, for, since their equations only differ in the sign of b^2, they have many properties in common which can be proved at the same time, by considering the sign of b^2 as indeterminate. We shall, in the following Articles, use the signs which apply to the ellipse. The reader may then obtain the corresponding formulæ for the hyperbola by changing the sign of b^2. We might infer the equation of the *tangent* to the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

from the general equation of a tangent given Art. 138; but the equation is so important, that we think it worth while to obtain it independently, by a method similar to that used Art. 84.

The equation of the line joining *any* two points is

$$\frac{y - y'}{x - x'} = \frac{y' - y''}{x' - x''}.$$

But if the two points be on the curve, we have

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 = \frac{x''^2}{a^2} + \frac{y''^2}{b^2};$$

hence

$$\frac{x'^2 - x''^2}{y'^2 - y''^2} = -\frac{a^2}{b^2},$$

and we have

$$\frac{y' - y''}{x' - x''} = -\frac{b^2}{a^2}\frac{x' + x''}{y' + y''}.$$

The equation of the chord is, therefore,

$$\frac{y - y'}{x - x'} = -\frac{b^2}{a^2}\frac{x' + x''}{y' + y''},$$

and that of the tangent,

$$\frac{y - y'}{x - x'} = -\frac{b^2}{a^2}\frac{x'}{y'};$$

or, reducing, and remembering that $x'y'$ satisfies the equation of the curve

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

a form easily remembered, from its similarity to the equation of the curve itself.

163. *To find the condition that any line* $\frac{x}{m} + \frac{y}{n} = 1$ *should touch the conic section* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

If we compare the equations

$$\frac{x}{m} + \frac{y}{n} = 1,$$

and

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

we find

$$\frac{x'}{a^2} = \frac{1}{m}, \text{ and } \frac{y'}{b^2} = \frac{1}{n};$$

but since

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

we have, for the required condition,

$$\frac{a^2}{m^2} + \frac{b^2}{n^2} = 1.$$

164. *To find the equation of the line joining the points of contact of tangents through any point* $(x'y')$.

We might infer, from Art. 139, that the equation of the *polar* is similar in form to the equation of the tangent, or we may prove it directly, as follows:

Let the co-ordinates of the point of contact of one of the tangents through $(x'y')$ be X, Y, then the equation of the tangent must be (Art. 163)

$$\frac{Xx}{a^2} + \frac{Yy}{b^2} = 1;$$

and, since this tangent passes through $(x'y')$ by hypothesis, we have

$$\frac{Xx'}{a^2} + \frac{Yy'}{b^2} = 1.$$

We see, therefore, that the co-ordinates of either point of contact must satisfy the equation

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1;$$

and, since this is the equation of a right line, it must represent the line joining them.

The polar of any point of the axis (whose $y' = 0$) is

$$\frac{xx'}{a^2} = 1,$$

that is, a line perpendicular to the axis at a distance $= \frac{a^2}{x'}$.

Hence, if several ellipses be described with the same axis major, the polar of any fixed point on the axis is the same for all.

CONJUGATE DIAMETERS.

165. We proved (Art. 151) that the equation of the curve, referred to any pair of conjugate diameters, is of the form

$$Ax^2 + Cy^2 = F,$$

which, if a', b', be the lengths of these diameters, may be written (as in Art. 155)

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$$

Now it can be proved, precisely as in Art. 162, that the equation of any tangent, referred to these axes, is

$$\frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 1,$$

and, as in Art. 164, that the equation of the polar of any point $(x'y')$ is of the same form. The polar of any point on the axis of x is, therefore,

$$\frac{xx'}{a'^2} = 1.$$

Hence, the polar of any point P is found by drawing a diameter through the point, taking $CP \cdot CP' =$ to the square of the semi-diameter, and then drawing through P′ a parallel to the conjugate diameter. This includes, as a particular case, the theorem proved already (Art. 140), viz.:

The tangent at the extremity of any diameter is parallel to the conjugate diameter.

166. The theorem just stated enables us easily to find the

equation, referred to the rectangular axes, of the diameter conjugate to that passing through any point on the curve $(x'y')$.

For the tangent at the point $(x'y')$ is (Art. 162)

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

Therefore, the conjugate diameter, which is a parallel to the tangent, drawn through the centre, is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0.$$

Let θ be the angle made with the axis of x by the original diameter, then $\tan\theta$ plainly $= \frac{y'}{x'}$, and if θ' be the angle made by the conjugate diameter, this equation shows that

$$\tan\theta' = -\frac{b^2 x'}{a^2 y'}.$$

Hence

$$\tan\theta \tan\theta' = -\frac{b^2}{a^2}.$$

This relation, connecting the angles made with the axis major by a pair of conjugate diameters, enables us at once to determine whether any given pair of diameters be conjugate or not.

The corresponding relation for the hyperbola is (see Art. 162)

$$\tan\theta \tan\theta' = \frac{b^2}{a^2}.$$

167. Since, in the ellipse, $\tan\theta \tan\theta'$ is negative, if one of the angles θ, θ', be acute (and, therefore, its tangent positive), the other must be obtuse (and, therefore, its tangent negative). Hence, *conjugate diameters in the ellipse lie on different sides of the axis minor* (which answers to $\theta = 90°$).

In the hyperbola, on the contrary, $\tan\theta \tan\theta'$ is positive, therefore, θ and θ' must be either both acute or both obtuse. Hence, *in the hyperbola, conjugate diameters lie on the same side of the conjugate axis.*

In the hyperbola, if $\tan\theta$ be less than $\frac{b}{a}$, $\tan\theta'$ must be greater than $\frac{b}{a}$, but (Art. 161) the diameter answering to the angle

whose tangent is $\frac{b}{a}$, is the asymptote which (by the same Article) separates those diameters which meet the curve from those which do not intersect it. Hence, *if one of two conjugate diameters meet a hyperbola in real points, the other will not.* Hence also it may be seen that each asymptote is its own conjugate.

168. The co-ordinates of the extremity of the diameter conjugate to that passing through $x'y'$, are found by combining the equation of the conjugate $\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0\right)$ with the equation of the curve.

If these co-ordinates be $x''y''$, it will be found that

$$\frac{x''}{a} = \mp\frac{y'}{b}, \text{ and } \frac{y''}{b} = \pm\frac{x'}{a};$$

for these co-ordinates satisfy the equation

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1,$$

and also the condition (Art. 166)

$$\frac{y''}{x''} = -\frac{b^2}{a^2}\frac{x'}{y'}.$$

It is often useful to express the lengths of a diameter (a'), and its conjugate (b'), in terms of the abscissa of the extremity of the diameter.

We have $$a'^2 = x'^2 + y'^2.$$

But $$y'^2 = \frac{b^2}{a^2}(a^2 - x'^2).$$

Hence $$a'^2 = b^2 + \frac{a^2 - b^2}{a^2}x'^2;$$

or (Art. 156) $$a'^2 = b^2 + e^2x'^2.$$

Again, we have $$b'^2 = x''^2 + y''^2 = \frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2,$$

or $$= a^2 - x'^2 + \frac{b^2}{a^2}x'^2;$$

hence $$b'^2 = a^2 - e^2x'^2.$$

From these values we have

$$a'^2 + b'^2 = a^2 + b^2,$$

or, *The sum of the squares of any pair of conjugate diameters of an ellipse is constant.*

169. In the hyperbola we must change the signs of b^2 and b'^2, and we get $$a'^2 - b'^2 = a^2 - b^2,$$

or, *The difference of the squares of any pair of conjugate diameters of a hyperbola is constant.*

If in the hyperbola we have $a = b$, its equation becomes

$$x^2 - y^2 = a^2,$$

and it is called an *equilateral* hyperbola.

The theorem just proved shows that *every diameter of an equilateral hyperbola is equal to its conjugate.*

The asymptotes of the equilateral hyperbola being given by the equation $$x^2 - y^2 = 0,$$

are at right angles to each other. Hence this hyperbola is often called a *rectangular* hyperbola.

The condition that the general equation of the second degree should represent an equilateral hyperbola is $A = -C$; for (Art. 74) this is the condition that the asymptotes ($Ax^2 + Bxy + Cy^2 = 0$) should be at right angles to each other; but if the hyperbola be *rectangular* it must be *equilateral*, since (Art. 161) the tangent of half the angle between the asymptotes $= \frac{b}{a}$; therefore, if this angle $= 45°$, we have $$b = a.$$

170. *To find the length of the perpendicular from the centre on the tangent.*

The length of the perpendicular from the origin on the line

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$

is (Art. 30)

$$\frac{1}{\sqrt{\left(\frac{x'^2}{a^4} + \frac{y'^2}{b^4}\right)}} = \frac{ab}{\sqrt{\left(\frac{b^2x'^2}{a^2} + \frac{a^2y'^2}{b^2}\right)}};$$

but we proved, Art. 168, that

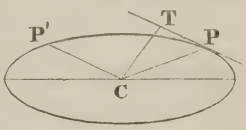
$$b'^2 = \frac{b^2 x'^2}{a^2} + \frac{a^2 y'^2}{b^2};$$

hence

$$p = \frac{ab}{b'}.$$

171. *To find the angle between any pair of conjugate diameters.*

The angle between the diameters is equal to the angle between either, and the tangent parallel to the other. Now



$$\sin \mathrm{CPT} = \frac{\mathrm{CT}}{\mathrm{CP}} = \frac{p}{a'}.$$

Hence

$$\sin\phi \text{ (or PCP')} = \frac{ab}{a'b'}.$$

The equation $a'b' \sin\phi = ab$ (a relation which may also be immediately inferred from Art. 156, III.) proves, that *the triangle formed by joining the extremities of conjugate diameters of an ellipse or hyperbola, has a constant area.*

172. The sum of the squares of any two conjugate diameters being constant, their rectangle is a maximum when they are equal, and, therefore, in this case, $\sin\phi$ is a minimum; hence the acute angle between the two *equal* conjugate diameters is less (and, consequently, the obtuse angle greater) than the angle between any other pair of conjugate diameters.

The length of the equal conjugate diameters is found by making $a' = b'$ in the equation $a'^2 + b'^2 = a^2 + b^2$, whence

$$a'^2 = \frac{a^2 + b^2}{2}.$$

Hence, in this case,

$$\sin\phi = \frac{2ab}{a^2 + b^2}.$$

The angle which either of the equiconjugate diameters makes with the axis of x, is found from the equation

$$\tan\theta \tan\theta' = -\frac{b^2}{a^2}.$$

by making $\tan\theta = -\tan\theta'$, for any two equal diameters make equal angles with the axis of x on opposite sides of it (Art. 156).

Hence

$$\tan\theta = \frac{b}{a}.$$

It follows, therefore, from Art. 161, that if an ellipse and hyperbola have the same axes in magnitude and position, then the asymptotes of the hyperbola will coincide with the equiconjugate diameters of the ellipse.

The general equation of an ellipse, referred to two conjugate diameters,

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

becomes $x^2 + y^2 = a'^2$, when $a' = b'$.

We see, therefore, that, by taking the equiconjugate diameters for axes, the equation of *any* ellipse may be put into the same form as the equation of the circle, $x^2 + y^2 = r^2$, but that in the case of the ellipse the angle between these axes will be *oblique.*

173. *To express the perpendicular from the centre on the tangent in terms of the angles which it makes with the axes.*

The equation of the tangent, expressed in terms of the perpendicular and the angle which it makes, is (Art. 30)

$$x\cos\alpha + y\sin\alpha = p.$$

This must be identical with the equation

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1;$$

hence

$$\frac{x'}{a^2} = \frac{\cos\alpha}{p}, \quad \text{or} \quad \frac{x'}{a} = \frac{a\cos\alpha}{p}.$$

In like manner

$$\frac{y'}{b} = \frac{b\sin\alpha}{p};$$

we have, therefore (Art. 155),

$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha.*$$

The equation of the tangent may, therefore, be written

$$x\cos\alpha + y\sin\alpha - \sqrt{(a^2\cos^2\alpha + b^2\sin^2\alpha)} = 0.$$

Hence, by Art. 31, the perpendicular from any point $(x'y')$ on the tangent is

$$x'\cos\alpha + y'\sin\alpha - \sqrt{(a^2\cos^2\alpha + b^2\sin^2\alpha)},$$

α being the angle which this perpendicular makes with the axis major.

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* In like manner $p^2 = a'^2\cos^2\alpha + b'^2\cos^2\beta$, α and β being the angles the perpendicular makes with any pair of conjugate diameters.

SUPPLEMENTAL CHORDS.

174. The chords which join the extremities of any diameter to any point on the curve are called *supplemental chords.*

Diameters parallel to any pair of supplemental chords are conjugate.

For if we consider the triangle formed by joining the extremities of any diameter AB to any point on the curve D; since, by elementary geometry, the line joining the middle points of two sides must be parallel to the third, the diameter bisecting AD will be parallel to BD, and the diameter bisecting BD will be parallel to AD. The same thing may be proved analytically, by showing that the product of the tangents of the angles made with the axis by AD and BD is $= -\frac{b^2}{a^2}$.

This property enables us to draw geometrically a pair of conjugate diameters making any angle with each other. For we have only to describe on any diameter a segment of a circle containing the given angle, then if we join the points where the circle meets the curve to the extremities of the assumed diameter, we obtain a pair of supplemental chords inclined at the given angle, and which (by the present Article) are parallel to a pair of conjugate diameters.

THE ECCENTRIC ANGLE.*

175. It is always advantageous to express the position of a point on a curve, if possible, by *a single independent* variable, rather than by the *two* co-ordinates $x'y'$. Thus (Art. 97), in the case of the circle, we found our formulæ simplified by the use of a subsidiary angle ϕ, where we wrote $r\cos\phi$ for x', and $r\sin\phi$ for y'. We shall find a similar substitution useful in discussing properties of the ellipse, and shall write

$$x' = a\cos\phi, \quad y' = b\sin\phi,$$

a substitution, evidently, consistent with the equation of the ellipse

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1.\dagger$$

* This section may be omitted on first reading the subject.

† The use of this angle ϕ occurred to me some years ago, as a particular case of the

The geometric meaning of the angle ϕ is easily explained.

If we describe a circle on the axis major as diameter, and produce the ordinate at P to meet the circle at Q, then the angle QCL = ϕ, for CL = CQ cos QCL, or $x' = a\cos\phi$, and PL = $\frac{b}{a}$ QL (Art. 157), or, since QL = CQ sin QCL = $a \sin\phi$, we have

$$y' = b \sin\phi.$$

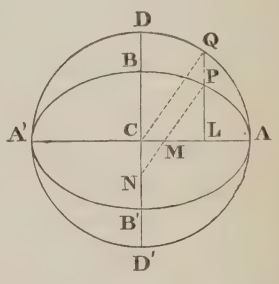
176. Some important consequences may be drawn from this construction.

If we draw through P a parallel PN to the radius CQ, then

$$PM : CQ :: PL : QL :: b : a,$$

but CQ = a, therefore PM = b.

PN parallel to CQ is, of course, = a.



Hence, if from any point of an ellipse a line = a be inflected to the minor axis, its intercept to the axis major = b.

Or, conversely, if a line MN, of a constant length, move about in the legs of a right angle, and be produced until MP be constant, the locus of P is an ellipse, whose axes are equal to MP and NP.

If the ordinate PQ were produced to meet the circle in the point Q′, it could be proved, in like manner, that a parallel through P to the radius CQ′ is cut into parts of a constant length. Hence, if a line of constant length move about in the legs of a right angle and be *cut* into constant parts at P, the locus of P will be an ellipse.

On this principle has been constructed an instrument for describing an ellipse by continued motion, called the *Elliptic Compasses*. CA, CD′, are two fixed rulers, MN a third ruler of a constant length, capable of sliding up and down between them, then a pencil fixed at any point of MN will describe an ellipse.

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methods given in Chap. XIII. It has, however, been already recommended by Mr. O'Brien in the *Cambridge Mathematical Journal*, vol. iv. p. 99, and has since been introduced by him, under the name here adopted, into his treatise on Plane Co-ordinate Geometry, p. 111.

If the pencil be fixed at the middle point of MN it will describe a circle.*

177. The consideration of the angle ϕ affords a simple method of constructing geometrically the diameter conjugate to a given one, for

$$\tan\theta = \frac{y'}{x'} = \frac{b}{a}\tan\phi.$$

Hence the relation

$$\tan\theta \tan\theta' = -\frac{b^2}{a^2}$$

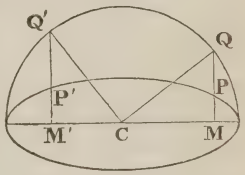
becomes

$$\tan\phi \tan\phi' = -1,$$

or

$$\phi - \phi' = 90°.$$

Hence we obtain the following construction for drawing the diameter conjugate to any given one. Let the ordinate at the given point P, when produced, meet the semicircle on the axis major at Q, join CQ, and erect CQ′ perpendicular to it, then the perpendicular let fall on the axis from Q′ will pass through P′, a point on the conjugate diameter.



Hence, too, can easily be found the co-ordinates of P′ given in Art. 168, for, since

$$\cos\phi' = \sin\phi, \text{ we have } \frac{x''}{a} = \frac{y'}{b},$$

and since

$$\sin\phi' = -\cos\phi, \text{ we have } \frac{y''}{b} = -\frac{x'}{a}.$$

From these values it appears that the areas of the triangles PCM, P′CM′, are equal.

178. The lengths of two conjugate semidiameters can be expressed in terms of the angle ϕ, for

$$a'^2 = x'^2 + y'^2,$$

or

$$a'^2 = a^2\cos^2\phi + b^2\sin^2\phi.$$

Again,

$$b'^2 = x''^2 + y''^2;$$

or, using the expressions for x'', y'', found in the last Article,

$$b'^2 = a^2\sin^2\phi + b^2\cos^2\phi.$$

* We shall give another proof of this theorem in Chap. XII. The proof here used is taken from O'Brien's *Co-ordinate Geometry*, p. 112.

Hence, as we saw before,

$$a'^2 + b'^2 = a^2 + b^2.$$

The equal conjugate diameters evidently answer to $\phi = 45°$.

The equation of any chord of the ellipse can be expressed in terms of ϕ and ϕ', as in Art. 97,

$$\frac{x}{a}\cos\tfrac{1}{2}(\phi+\phi') + \frac{y}{b}\sin\tfrac{1}{2}(\phi+\phi') = \cos\tfrac{1}{2}(\phi-\phi'),$$

and that of the tangent

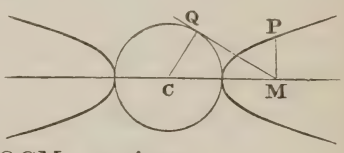
$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1.$$

179. The methods of the last four Articles do not apply to the hyperbola. For the hyperbola, however, we may substitute

$$x' = a\sec\phi, \qquad y' = b\tan\phi,$$

since

$$\left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{b}\right)^2 = 1.$$



This angle may be represented geometrically, by drawing a tangent MQ from the foot of the ordinate M to the circle described on the transverse axis, then the angle QCM = ϕ, since

$$\text{CM} = \text{CQ}\sec\text{QCM}.$$

We have also QM = $a\tan\phi$, but PM = $b\tan\phi$.

Hence, if from the foot of any ordinate of a hyperbola we draw a tangent to the circle described on the transverse axis, this tangent is in a constant ratio to the ordinate.

180.* Since the equation of the conjugate hyperbola is

$$\left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1,$$

any point on the conjugate hyperbola may be expressed by

$$y'' = b\sec\phi', \text{ and } x'' = a\tan\phi'.$$

Now if θ be the angle made by any diameter with the axis of x, we have

$$\tan\theta = \frac{y'}{x'} = \frac{b}{a}\sin\phi.$$

* This Article is taken from a paper by Mr. Turner in the *Cambridge and Dublin Math. Jour.*, vol. i. p. 122.

In like manner $$\tan\theta' = \frac{y''}{x''} = \frac{b}{a}\frac{1}{\sin\phi'};$$

hence the relation connecting two conjugate diameters

$$\tan\theta \tan\theta' = \frac{b^2}{a^2}$$

becomes $$\sin\phi = \sin\phi';$$

or, simply, $$\phi = \phi'.$$

THE NORMAL.

181. A line drawn through any point of the curve perpendicular to the tangent at that point is called the *Normal*.

Its equation is easily found by Art. 39, for it is a line drawn through $(x'y')$ perpendicular to the line whose equation is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1;$$

its equation is, therefore,

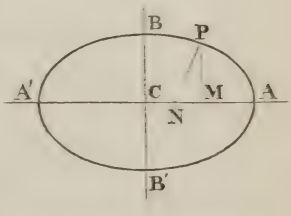
$$\frac{x'}{a^2}(y - y') = \frac{y'}{b^2}(x - x'),$$

or $$\frac{a^2x}{x'} - \frac{b^2y}{y'} = c^2,$$

c^2 being used, as in Art. 156, to denote $a^2 - b^2$.

Hence we can find the portion CN intercepted by the normal on either axis; for, making $y = 0$ in the equation just given, we find

$$x = \frac{c^2}{a^2}x', \text{ or } x = e^2x',$$



e being $= \frac{c}{a}$, as in Art. 156.

We can thus draw a normal to an ellipse from any point on the axis, for given CN we can find x', the ordinate of the point through which the normal is drawn.

The circle may be considered as an ellipse whose eccentricity $= 0$, since $c^2 = a^2 - b^2 = 0$. The intercept CN, therefore, is constantly $= 0$ in the case of the circle, or *every normal to a circle passes through its centre.*

182. The portion MN intercepted on the axis between the normal and ordinate is called the *Subnormal.* Its length is, by the last Article,

$$x' - \frac{c^2}{a^2}x' = \frac{b^2}{a^2}x'.$$

The normal, therefore, cuts the abscissa into parts which are in a constant ratio.

If a tangent drawn at the point P cut the axis in T, the intercept MT is, in like manner, called the *Subtangent.*

Since the whole length $CT = \frac{a^2}{x'}$ (Art. 165), the subtangent

$$= \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'}.$$

The length of the *normal* can also be easily found. For

$$PN^2 = PM^2 + NM^2 = y'^2 + \frac{b^4}{a^4}x'^2 = \frac{b^2}{a^2}\left(\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2\right),$$

but if b' be the semidiameter conjugate to CP, then (Art. 168)

$$\frac{b^2}{a^2}x'^2 + \frac{a^2}{b^2}y'^2 = b'^2.$$

Hence the length of the normal $PN = \frac{bb'}{a}$.

If the normal be produced to meet the axis minor it can be proved, in like manner, that its length $= \frac{ab'}{b}$. Hence, *the rectangle under the segments of the normal is equal to the square of the semi-conjugate diameter.*

Again, we found (Art. 172) that the perpendicular from the centre on the tangent $= \frac{ab}{b'}$. Hence, *the rectangle under the normal and the perpendicular from the centre on the tangent, is constant and equal to the square of the semiaxis.*

We can thus express the normal in terms of the angles it makes with the axis, for

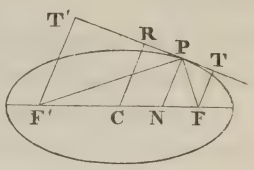
$$PN = \frac{b^2}{p} = \frac{b^2}{\sqrt{(a^2\cos^2\alpha + b^2\sin^2\alpha)}} \text{ (Art. 173)}; = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2\alpha)}}.$$

THE FOCI.

183. If on the axis major of an ellipse we take two points equidistant from the centre, whose common distance

$$= \pm \sqrt{(a^2 - b^2)}, \text{ or } = \pm c,$$

these points are called the *foci* of the curve.



The foci of an hyperbola are two points on the transverse axis, at a distance from the centre still $= \pm c$, c being in the hyperbola

$$= \sqrt{(a^2 + b^2)}.$$

To express the distance of any point on an ellipse from the focus.

Since the co-ordinates of one focus are $(x = +c,\ y = 0)$, the square of the distance of any point from it

$$= (x' - c)^2 + y'^2 = x'^2 + y'^2 - 2cx' + c^2.$$

But (Art. 168)

$$x'^2 + y'^2 = b^2 + e^2x^2, \text{ and } b^2 + c^2 = a^2.$$

Hence $$\text{FP}^2 = a^2 - 2cx + e^2x;$$

and, recollecting that $c = ae$, we have

$$\text{FP} = a - ex.$$

(We reject the value $(ex - a)$ obtained by giving the other sign to the square root. For, since x is less than a, and e less than 1, the quantity $ex - a$ is constantly negative, and, therefore, does not concern us, as we are now considering, not the direction, but the absolute magnitude of the radius vector FP).

We have, similarly, the distance from the other focus

$$\text{F'P} = a + ex,$$

since we have only to write $-c$ for $+c$ in the preceding formulæ.

Hence $$\text{FP} + \text{F'P} = 2a,$$

or, *The sum of the distances of any point on an ellipse from the foci is constant and equal to the axis major.*

184. In applying the preceding proposition to the hyperbola we obtain the same value for FP^2, but in extracting the square root we must change the sign in the value of FP, for in the hyperbola x is greater than a, and e is greater than 1.

Hence, $a - ex$ is constantly negative; the absolute magnitude, therefore, of the radius vector is

$$\text{FP} = ex - a.$$

In like manner, $F'P = ex + a.$

Hence $F'P - FP = 2a.$

Therefore, *in the hyperbola, the* difference *of the focal radii is constant, and equal to the transverse axis.*

For both curves the rectangle *under the focal radii* $= a^2 - e^2x^2$, that is (Art. 168), *is equal to the square of the semiconjugate diameter.*

185. The reader may prove the converse of the above results by seeking the locus of the vertex of a triangle, if the base and either sum or difference of sides be given.

Taking the middle point of the base ($= 2c$) for origin, the equation is

$$\sqrt{\{y^2 + (c + x)^2\}} \pm \sqrt{\{y^2 + (c - x)^2\}} = 2a,$$

which, when cleared of radicals, becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Now, if the *sum* of the sides be given, since the sum must always be greater than the base, a is greater than c, therefore the co-efficient of y^2 is positive, and the locus an *ellipse.*

If the *difference* be given, a is less than c, the coefficient of y^2 is negative, and the locus an *hyperbola.*

186. *To find the length of the perpendicular from the focus on the tangent.*

The length of the perpendicular from the focus $(+c, 0)$ on the line $\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1\right)$ is, by Art. 31,

$$= \frac{1 - \frac{cx'}{a^2}}{\sqrt{\left(\frac{x'^2}{a^4} + \frac{y'^2}{b^4}\right)}};$$

but, Art. 170, $$\sqrt{\left(\frac{x'^2}{a^4} + \frac{y'^2}{b^4}\right)} = \frac{b'}{ab}.$$

Hence $$FT = \frac{b}{b'}(a - ex) = \frac{b}{b'} FP.$$

Likewise, $$F'T' = \frac{b}{b'}(a + ex) = \frac{b}{b'} F'P.$$

Hence $FT \cdot F'T' = b^2$ (since $a^2 - e^2x^2 = b^2$),

or, *The rectangle under the focal perpendiculars on the tangent is constant, and equal to the square of the semiaxis minor.*

This property applies equally to the ellipse and the hyperbola.

187. Some important consequences may be drawn from the value of the perpendicular just found.

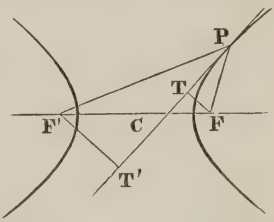
For we had $$FT = \frac{b}{b'} FP, \text{ or } \frac{FT}{FP} = \frac{b}{b'};$$

but $$\frac{FT}{FP} = \sin FPT.$$

Hence the sine of the angle which the focal radius vector makes with the tangent $= \frac{b}{b'}$.

We find, in like manner, the same value for $\sin F'PT'$, the sine of the angle which the other focal radius vector makes with the tangent. Hence *the focal radii make equal angles with the tangent.*

This property is true both for the ellipse and hyperbola, and, on looking at the figures, it is evident that the tangent to the ellipse is the *external* bisector of the angle between the focal radii, and the tangent to the hyperbola the *internal* bisector.



Hence, *if an ellipse and hyperbola, having the same foci, pass through the same point, they will cut each other at right angles*, that is to say, the tangent to the ellipse at that point will be at right angles to the tangent to the hyperbola.

188. The normal, being perpendicular to the tangent, is the *internal* bisector of the angle between the focal radii in the case of the ellipse, and the *external* bisector in the case of the hyperbola.

We can give an independent proof of this, by showing that it cuts the distance between the foci into parts which are in the ratio of the focal radii (Euc. VI. 3), for the distance of the foot of the normal from the centre is (Art. 183) $= e^2x'$. Hence its distances from the foci are $c + e^2x'$ and $c - e^2x'$, quantities which are evidently e times $a + ex'$ and $a - ex'$.

From the preceding theorem can be derived a simple solution of the problem, to draw a normal to the ellipse from any point on the axis minor, for the circle through the given point, and the two foci, will meet the curve at the point whence the normal is to be drawn.

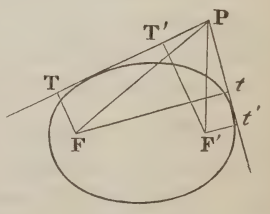
It follows from the preceding Article, that if a line be drawn through the centre parallel to either focal radius vector, and terminated by the tangent, its length will be $= a$, for the perpendicular from the centre on the tangent $= \frac{ab}{b'}$ (Art. 170), but this perpendicular, divided by sin FPT $\left(= \frac{b}{b'}\right)$, is the length of the required parallel.

189. Another important consequence may be deduced from the theorem (of Art. 186), that the rectangle under the focal perpendiculars on the tangent is constant.

For, if we take any two tangents, we have

$$FT \cdot F'T' = Ft \cdot F't', \text{ or } \frac{FT}{Ft} = \frac{F't'}{F'T'};$$

but $\frac{FT}{Ft}$ is the ratio of the sines of the parts into which the line FP divides the angle at P, and $\frac{F't'}{F'T'}$ is the ratio of the sines of the parts into which F'P divides the same angle; we have, therefore, the angle TPF $= t'$PF'.



If we conceive a conic section to pass through P, having F and F' for foci, it was proved in Art. 187, that the tangent to it must be equally inclined to the lines FP, F'P; it follows, therefore, from the present Article, that it must be also equally inclined to PT, Pt; hence we derive a useful theorem, that *if through any point* (P) *of a conic section we draw tangents* (PT, Pt) *to a* confocal *conic section, these tangents will be equally inclined to the tangent at* P.

190. *To find the locus of the foot of the perpendicular let fall from either focus on the tangent.*

From Art. 173 the polar equation of the locus can immediately be found, for since the perpendicular expressed in terms of the angles it makes with the axis is

$$x' \cos\alpha + y' \sin\alpha - \sqrt{(a^2 \cos^2\alpha + b^2 \sin^2\alpha)},$$

the perpendicular from the focus $(+c, 0)$ is

$$c \cos\alpha - \sqrt{(a^2 \cos^2\alpha + b^2 \sin^2\alpha)},$$

and, therefore, the polar equation of the locus is

$$\rho = c \cos\alpha - \sqrt{(a^2 \cos^2\alpha + b^2 \sin^2\alpha)},$$

or $$\rho^2 - 2c\rho \cos\alpha + c^2 \cos^2\alpha = a^2 \cos^2\alpha + b^2 \sin^2\alpha,$$

or $$\rho^2 - 2c\rho \cos\alpha = b^2.$$

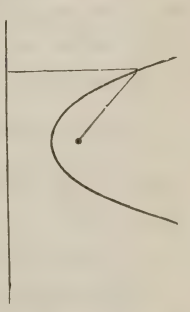
This (Art. 94) is the polar equation of a circle whose centre is on the axis of x, at a distance from the focus $= c$; the circle is, therefore, concentric with the curve. The radius of the circle is, by the same Article, $= a$.

Hence, *if we describe a circle having for diameter the transverse axis of an ellipse or hyperbola, the perpendicular from the focus will meet the tangent on the circumference of this circle.*

Or, conversely, *if from any point* F (see figure, p. 163) *we draw a radius vector* FT *to a given circle, and draw* TP *perpendicular to* FT, *the line* TP *will always touch a conic section having* F *for its focus, which will be an ellipse or hyperbola, according as* F *is within or without the circle.*

It may be inferred from Art. 188, that the line CT, whose length $= a$, is parallel to the focal radius vector F′P.

191. The polar of either focus is called the *directrix* of the conic section. The directrix must, therefore (Art. 165), be a line perpendicular to the axis major at a distance from the centre $= \pm \frac{a^2}{c}$.



Knowing the distance of the directrix from the centre we can find its distance from any point on the curve. It must be equal to

$$\frac{a^2}{c} - x', \text{ or } = \frac{a}{c}(a - ex') = \frac{1}{e}(a - ex').$$

But the distance of any point on the curve from the focus $= a - ex$ (Art. 183).

Hence we obtain the important property, that *the distance of any point on the curve from the focus is in a constant ratio to its distance from the directrix*, viz., as e to 1.

In the ellipse e is less than 1, and the distance from the focus is *less* than the distance from the directrix. In the hyperbola e is greater than 1, and the distance from the focus is *greater* than the distance from the directrix.

We shall show, in the next chapter, that the parabola also has a focus, and that in this case the distance of any point from the focus is *equal* to the distance from the directrix.

192. *If any chord be drawn through the focus, tangents at its extremities will intersect on the directrix* (this follows from the definition of the directrix and Art. 141); *and the line joining the point of intersection to the focus will be perpendicular to the chord.*

We shall deduce this as a particular case of a more general theorem.

The equation of the polar of any point $\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1 = 0\right)$ being perfectly similar to the equation of the tangent, the equation of the perpendicular on the polar from the point $x'y'$ must be similar to the equation of the normal,

$$\frac{a^2x}{x'} - \frac{b^2y}{y'} = c^2 \text{ (Art. 181)},$$

and the intercept it makes on the axis of x is, as in Art. 181, $= e^2x'$. If x', therefore, be constant, the intercept will be constant, that is, *If a point be taken anywhere on a fixed perpendicular to the axis, the perpendicular from it on its polar will pass through a fixed point on the axis.*

If the point be taken anywhere on the directrix then x' will $= \frac{a^2}{c}$, and, therefore, $e^2x' = c$. Hence, the perpendicular from any point of the directrix on its polar will pass through the focus. Q. E. D.*

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* The method used in this Article was communicated to me by the Rev. W. D. Sadleir, together with several other theorems, which will be given in Chap. XII., also deduced from the analogy between the equations of the polar and the tangent.

193. *To find the polar equation of the ellipse or hyperbola, the focus being the pole.*

This might be obtained by transforming the rectangular equation to the focus as origin, and then substituting $\rho \cos\theta$ and $\rho \sin\theta$ for x and y; or it may be derived immediately from the value of the focal radius vector (Art. 183)

$$\rho = a - ex'.$$

For x' (being measured from the centre) $= \rho \cos\theta + c$.

Hence $$\rho = a - e\rho \cos\theta - ec,$$

or $$\rho = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{b^2}{a} \cdot \frac{1}{1+e\cos\theta}.$$

The double ordinate at the focus is called the *parameter;* its half is found by making $\theta = 90°$ in the equation just given, to be $= \frac{b^2}{a} = a(1-e^2)$. The parameter is commonly denoted by the letter p. Hence the equation is often written

$$\rho = \frac{p}{2} \cdot \frac{1}{1+e\cos\theta}.$$

The parameter is also called the *Latus Rectum.*

194. Several useful inferences may be drawn from the polar equation.

If the radius vector FP, when produced backwards through the focus, meet the curve again in P′, then FP being $\frac{p}{2} \cdot \frac{1}{1+e\cos\theta}$, FP′, which answers to $(\theta + 180°)$, will $= \frac{p}{2} \cdot \frac{1}{1-e\cos\theta}$.

Hence $$\frac{1}{\text{FP}} + \frac{1}{\text{FP}'} = \frac{4}{p},$$

or, *The harmonic mean between the segments of a focal chord is constant, and equal to the semiparameter.*

This may otherwise be stated

$$\frac{\text{FP} \cdot \text{FP}'}{\text{FP} + \text{FP}'} = \frac{1}{p},$$

or, *The rectangle under the segments of a focal chord is to the whole chord in a constant ratio.*

The rectangle under the segments

$$= \frac{p^2}{4} \cdot \frac{1}{1 - e^2 \cos^2 \theta} = \frac{b^4}{a^2} \cdot \frac{1}{1 - e^2 \cos^2 \theta},$$

and the whole chord (FP + FP')

$$= p \cdot \frac{1}{1 - e^2 \cos^2 \theta} = \frac{2b^2}{a} \cdot \frac{1}{1 - e^2 \cos^2 \theta}.$$

195. It will be remembered that the length of a semidiameter making an angle θ with the transverse axis is (Art. 156)

$$R^2 = \frac{b^2}{1 - e^2 \cos^2 \theta}.$$

Hence the length of the chord just given $= \frac{2R^2}{a}$, or

Any focal chord is a third proportional to the transverse axis and the parallel diameter.

Again, we know that the sum of the squares of two conjugate diameters is constant (Art. 168). Hence, *the sum of two focal chords drawn parallel to two conjugate diameters is constant.*

Or, again, we can prove that *the sum of the reciprocals of the squares of two semidiameters at right angles to each other is constant.*

For one reciprocal $= \frac{1 - e^2 \cos^2 \theta}{b^2}$, and the other $= \frac{1 - e^2 \sin^2 \theta}{b^2}$.

Hence, therefore, *the sum of the reciprocals of two focal chords at right angles to each other is constant.*

196. While speaking of polar co-ordinates we may mention, that in any curve the portion intercepted by the tangent on a perpendicular to the radius vector drawn through the pole is called the *polar subtangent.* Hence the theorem proved in Art. 192 may be stated thus: *In the ellipse and hyperbola, the focus being the pole, the locus of the extremity of the polar subtangent is the directrix.* We shall prove, in the next chapter, that this is equally true for the parabola.

197. The equation of the ellipse, referred to the vertex, is

$$\frac{(x - a)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

or

$$y^2 = \frac{2b^2}{a} x - \frac{b^2}{a^2} x^2 = px - \frac{b^2}{a^2} x^2.$$

Hence, in the ellipse, the square of the ordinate *is less than* the rectangle under the parameter and abscissa.

The equation of the hyperbola is found in like manner,

$$y^2 = px + \frac{b^2}{a^2}x^2.$$

Hence, in the hyperbola, the square of the ordinate *exceeds* the rectangle under the parameter and abscissa.

We shall show, in the next chapter, that in the parabola these quantities are *equal*.

It was from this property that the names *parabola*, *hyperbola*, and *ellipse*, were first given.

THE ASYMPTOTES.

198. We have hitherto discussed properties common to the ellipse and the hyperbola. There is, however, one class of properties of the hyperbola which have none corresponding to them in the ellipse, those, namely, depending on the asymptotes, which in the ellipse are imaginary.

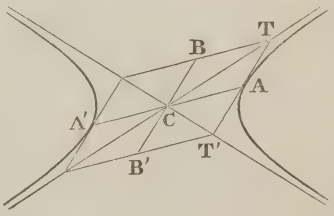
We saw that the equations of the asymptotes were always obtained by putting the highest powers of the variables = 0, the centre being the origin. Thus the equation of the curve, referred to any pair of conjugate diameters, being

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1,$$

that of the asymptotes is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0, \text{ or } \frac{x}{a'} - \frac{y}{b'} = 0, \text{ and } \frac{x}{a'} + \frac{y}{b'} = 0.$$

Hence the asymptotes are parallel to the diagonals of the parallelogram, whose adjacent sides are any pair of conjugate semi-diameters. For, the equation of CT is $\frac{y}{x} = \frac{b'}{a'}$, and must, therefore, coincide with one asymptote, while the equation of AB $\left(\frac{x}{a'} + \frac{y}{b'} = 1\right)$ is parallel to the other (see Art. 161).



Hence, given any two conjugate diameters, we can find the

asymptotes; or, given the asymptotes, we can find the diameter conjugate to any given one; for we have only to draw AO parallel to one asymptote, to meet the other, and produce it equal to itself, when we find B, the extremity of the conjugate diameter.

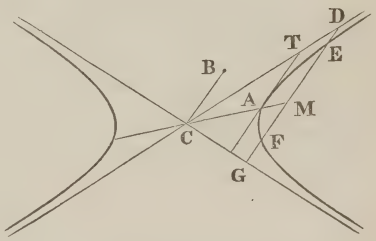
199. *The portion between the asymptotes of any tangent to a hyperbola is bisected at the curve, and is equal to the conjugate diameter.*

This appears at once from the last Article, where we have proved $AT = b' = AT'$; or, directly, taking for axes the diameter through the point and its conjugate, the equation of the asymptotes is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0.$$

Hence, if we take $x = a'$, we have $y = \pm b'$; but the tangent at A being parallel to the conjugate diameter, this value of the ordinate is the intercept on the tangent.

200. *If any line cut an hyperbola, the portions* DE, FG, *intercepted between the curve and its asymptotes, are equal.*



For, if we take for axes a diameter parallel to DG and its conjugate, it appears from the last Article, that the portion DG is bisected by the diameter; so is also the portion EF; hence DE = FG.

The lengths of these lines can immediately be found, for, from the equation of the asymptotes $\left(\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0\right)$, we have

$$y\,(= \mathrm{DM} = \mathrm{MG}) = \pm \frac{b'}{a'} x.$$

Again, from the equation of the curve

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1,$$

we have

$$y\,(= \mathrm{EM} = \mathrm{FM}) = \pm b' \sqrt{\left(\frac{x^2}{a'^2} - 1\right)}.$$

Hence $$\mathrm{DE}\,(=\mathrm{FG}) = b'\left\{\frac{x}{a'} - \sqrt{\left(\frac{x^2}{a'^2} - 1\right)}\right\},$$

and $$\mathrm{DF}\,(=\mathrm{EG}) = b'\left\{\frac{x}{a'} + \sqrt{\left(\frac{x^2}{a'^2} - 1\right)}\right\}.$$

201. From these equations it at once follows, that *the rectangle* DE . DF *is constant, and* $= b'^2$. Hence, the greater DF is, the smaller will DE be. Now it is evident, that the further from the centre we draw DF the greater will it be, and that by taking x sufficiently large, we can make DF $\left[= b'\left\{\frac{x}{a'} + \sqrt{\left(\frac{x^2}{a'^2} - 1\right)}\right\}\right]$ greater than any assigned quantity. Hence, *the further from the centre we draw any line, the less will be the interval between the curve and its asymptote, and by increasing the distance from the centre, we can make this interval less than any assigned quantity.*

202. It appears from Art. 161, that if the equations of the asymptotes be $\alpha = 0$, $\beta = 0$, the equation of the curve must be of the form $\alpha\beta = k^2$. If the asymptotes be chosen for axes of co-ordinates, their equations are $x = 0$, $y = 0$, and the equation of the curve must be of the form $xy = k^2$, as we saw (Art. 161).

The equation of the curve, in this form, may be employed with advantage in investigating properties of the asymptotes.

The geometrical meaning of this equation evidently is, that *the area of the parallelogram formed by the co-ordinates is constant.*

203. If the equation be given in the form $xy = k^2$, we can easily form the equation of any chord or of any tangent.

We have $$y' = \frac{k^2}{x'}, \text{ and } y'' = \frac{k^2}{x''};$$

therefore, $$y' - y'' = -\frac{k^2(x' - x'')}{x'x''};$$

the equation of a chord is, therefore,

$$\frac{y - y'}{x - x'} = -\frac{k^2}{x'x''},$$

which may be written (since $k^2 = x'y' = x''y''$)

$$y'x + x''y = k^2 + y'x''.$$

Making $x' = x''$ and $y' = y''$, we find the equation of the tangent,

$$x'y + y'x = 2k^2,$$

or (writing $x'y'$ for k^2)

$$\frac{x}{x'} + \frac{y}{y'} = 2.$$

From this form it appears that the intercepts made on the asymptotes by any tangent $= 2x'$ and $2y'$; their rectangle is, therefore, $4k^2$. Hence, *the triangle which any tangent forms with the asymptotes has a constant area, and is equal to double the area of the parallelogram formed by the co-ordinates.*

204. It is desirable to express the quantity k^2 in terms of the lengths of the axes of the curve.

Since the axis bisects the angle between the asymptotes, the co-ordinates of its vertex are found, by putting $x = y$ in the equation $xy = k^2$, to be $x = y = k$.

Hence, if θ be the angle between the axis and the asymptote,

$$a = 2k \cos \theta$$

(since a is the base of an isosceles triangle whose sides $= k$ and base angle $= \theta$), but (Art. 161)

$$\cos \theta = \frac{a}{\sqrt{(a^2 + b^2)}};$$

hence

$$k = \frac{\sqrt{(a^2 + b^2)}}{2}.$$

And the equation of the curve, referred to its asymptotes, is

$$xy = \frac{a^2 + b^2}{4}.$$

205. The perpendicular from the focus on the asymptote is equal to the conjugate axis b.

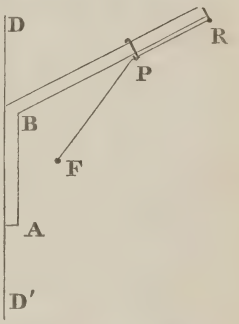
For it is CF $\sin \theta$, but CF $= \sqrt{(a^2 + b^2)}$, and $\sin \theta = \dfrac{b}{\sqrt{(a^2 + b^2)}}$ (Art. 161).

This might be deduced as a particular case of the property, that the product of the perpendiculars from the focus on any tangent is constant, and $= b^2$. For the asymptote may be considered as a tangent, whose point of contact is at an infinite distance (Art. 161), and the perpendiculars from the foci on it are evidently equal to each other.

206. *The distance of the focus from any point on the curve is equal to the length of a line drawn through the point parallel to an asymptote to meet the directrix.*

For the distance from the focus is e times the distance from the directrix (Art. 191), and the distance from the directrix is to the length of the parallel line as $\cos\theta \left(= \frac{1}{e}, \text{ Art. } 161\right)$ is to 1.

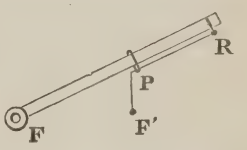
Hence has been derived a method of describing the hyperbola by continued motion. A ruler ABR, bent at B, slides along the fixed line DD′; a thread of a length = RB is fastened at the two points R and F, while a ring at P keeps the thread always stretched, then as the ruler is moved along, the point P will describe an hyperbola, of which F is a focus, DD′ a directrix, and BR parallel to an asymptote, since PF must always = PB.



207. Besides this method of mechanically describing an hyperbola, and the method of describing an ellipse given Art. 176, there is another depending on the fundamental property that the sum or difference of the focal distances of any point on the curve is constant.

If the extremities of a thread be fastened at two fixed points F and F′, it is plain that a pencil moved about so as to keep the thread always stretched, will describe an ellipse whose foci are F and F′, and whose axis major is equal to the length of the thread.

In order to describe an hyperbola, let a ruler be fastened at one extremity (F), and capable of moving round it, then if a thread, fastened to a fixed point F′, and also to a fixed point on the ruler (R), be kept stretched by a ring at P, as the ruler is moved round, the point P will describe an hyperbola; for, since the sum of F′P and PR is constant, the difference of FP and F′P will be constant.



CHAPTER XI.

THE PARABOLA.

208. The equation of the second degree, we saw (Art. 129), will represent a parabola, when the first three terms form a perfect square, or when the equation is of the form

$$(ax + by)^2 + Dx + Ey + F = 0.$$

We saw that we could not transform this equation to any finite point so as to make the coefficients of x and y both vanish. The form of the equation, however, points at once to another method of simplifying it.

We know (Art. 31) that the quantity $Dx + Ey + F$ is proportional to the length of the perpendicular from the point (xy) on the right line whose equation is $Dx + Ey + F = 0$; and, in like manner, the quantity $ax + by$ is proportional to the perpendicular on the line $ax + by = 0$.

Hence if we construct the two lines whose equations are

$$ax + by = 0, \qquad Dx + Ey + F = 0,$$

the equation of the curve asserts that the square of the perpendicular from any point of the curve on the first line is in a constant ratio to the perpendicular on the second line.

Now if we transform our axes, and make the line $ax + by$ our new axis of x, and $Dx + Ey + F = 0$ our new axis of y, then our new y will, of course, be proportional to the perpendicular on the line $ax + by$, and our new x to the perpendicular on $Dx + Ey + F = 0$, and the transformed equation must be of the form

$$y^2 = px.$$

It is evident that our new origin is a point on the curve, and since for every value of x we have two equal and opposite values of y, our new axis of x will be a diameter, and our new axis of y parallel to its ordinates. But the ordinate drawn at the extremity of any diameter is (Art. 140) a tangent to the curve, therefore,

the new axis of y is the tangent at the origin. Hence, if we are given the equation of the parabola in the form

$$(ax + by)^2 + Dx + Ey + F = 0,$$

the equation $ax + by = 0$ represents the diameter passing through the origin, and the equation $Dx + Ey + F = 0$ represents the tangent at the point where this diameter meets the curve. And the equation of the curve, referred to a diameter and tangent at its extremity as axes, is of the form

$$y^2 = px.$$

209. Though we have transformed the equation of the parabola into a very simple form, yet our new axes have the inconvenience of not being in general rectangular. We shall prove, however, that it is possible to transform the equation into this form, still retaining the axes rectangular.

If we introduce an arbitrary constant k, the equation

$$(ax + by)^2 + Dx + Ey + F = 0$$

will be found to be equivalent to the equation

$$(ax + by + k)^2 + (D - 2ak)\,x + (E - 2bk)\,y + F - k^2 = 0.$$

Hence, as in the last Article,

$$ax + by + k = 0$$

is the equation of a diameter, and

$$(D - 2ak)\,x + (E - 2bk)\,y + F - k^2 = 0$$

of the tangent at its extremity. (This confirms our proof (Art. 135) that all the diameters of the parabola are parallel).

Now, the condition that these two lines should be perpendicular is (Art. 38)

$$a\,(D - 2ak) + b\,(E - 2bk) = 0.$$

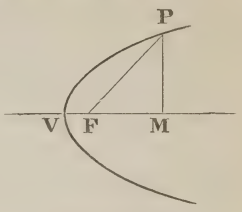
Hence

$$k = \frac{aD + bE}{2\,(a^2 + b^2)}.$$

Since we get a simple equation to determine the particular value of k, which would make the new axes rectangular, there is *one* diameter whose ordinates cut it perpendicularly, and this diameter is called the *axis* of the curve. And we see, as in the last Article, that if we take for axes, this diameter $ax + by + k$, and the perpendicular tangent $(D - 2ak)\,x + (E - 2bk)\,y + F - k^2 = 0$, the transformed equation must be of the form

$$y^2 = px.$$

210. From the equation $y^2 = px$ we can at once perceive the figure of the curve. It must be symmetrical on both sides of the axis of x, since every value for x gives two equal and opposite for y. None of it can lie on the negative side of the origin, since if we make x negative y will be imaginary; and as we give increasing positive values to x, we obtain increasing values for y. Hence the figure of the curve is that here represented.



Although the parabola resembles the hyperbola in having infinite branches, yet there is an important difference between the nature of the infinite branches of the two curves. Those of the hyperbola, we saw, tend ultimately to coincide with two diverging right lines; but this is not true for the parabola, since, if we seek the points where any right line ($x = ky + l$) meets the parabola ($y^2 = px$), we obtain the quadratic

$$y^2 - pky - pl = 0,$$

whose roots can never be infinite as long as k and l are finite.

We proved before, that the diameters are the only lines which pass through a point of the curve at infinity, and if we seek the point in which any diameter ($y = m$) meets the curve, we find

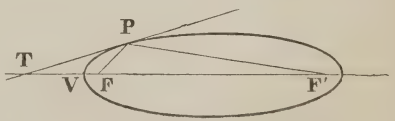
$$x = \frac{m^2}{p}.$$

Now, although this value increases as m increases, yet it will never become infinite as long as m is finite; hence, though the curve tends more and more nearly to become parallel to the diameters, yet it does not actually become so at any finite distance.

211. The figure of the parabola may be more clearly conceived from the following theorem.

If we suppose one vertex and focus of an ellipse given, while its axis major increases without limit, the curve will ultimately become a parabola.

The equation of the ellipse, referred to its vertex, is (Art. 197)



$$y^2 = \frac{2b^2}{a}x - \frac{b^2}{a^2}x^2.$$

We wish to express b in terms of the distance VF ($= m$), which we suppose fixed. We have $m = a - \sqrt{(a^2 - b^2)}$ (Art. 183), whence $b^2 = 2am - m^2$, and the equation becomes

$$y^2 = \left(4m - \frac{2m^2}{a}\right)x - \left(\frac{2m}{a} - \frac{m^2}{a^2}\right)x^2.$$

Now, if we suppose a to become infinite, all but the first term of the right-hand side of the equation will vanish, and the equation becomes $y^2 = 4mx$,

the equation of a parabola.

Hence we see that the focus and vertex of an ellipse being given, while the axis major is indefinitely increased, the parameter $\left(= \frac{2b^2}{a}, \text{ Art. } 193\right)$ will remain finite, and $= 4m$.

Hence if the equation of the parabola be given in the form $y^2 = px$, the quantity p is called the principal parameter.

A parabola may also be considered as an ellipse whose eccentricity is equal to 1. For $e^2 = 1 - \frac{b^2}{a^2}$. Now we saw that $\frac{b^2}{a^2}$, which is the co-efficient of x^2 in the preceding equation, vanished as we supposed a increased according to the prescribed conditions; hence e^2 becomes finally $= 1$.

THE TANGENT.

212. The equation of any chord of the parabola can be easily obtained. For, since $y'^2 = px'$ and $y''^2 = px''$, we have

$$y'^2 - y''^2 = p(x' - x''), \text{ and } \frac{y' - y''}{x' - x''} = \frac{p}{y' + y''},$$

and the equation of the chord is

$$\frac{y - y'}{x - x'} = \frac{p}{y' + y''},$$

or $$(y' + y'')y - px - y'y'' = 0.$$

The equation of the tangent is found from this, by supposing $y' = y''$, or (remembering that $y'^2 = px'$) is $2y'y = p(x + x')$.

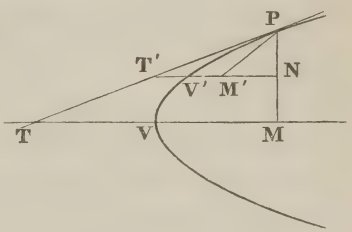
If we seek the point where the tangent meets the axis, we obtain $x = -x'$, or TM (which is called the *subtangent*) is bisected at the vertex.

We saw that if the oblique axes were any diameter and a tangent through its vertex, the equation of the parabola was still

$$y^2 = p'x.$$

The equations of the chord and tangent remain the same, and it will be equally true that the subtangent is = twice the abscissa.

This Article enables us, therefore, to draw a tangent at any point on the parabola, since we have only to take TV = VM and join PT; or again, having found this tangent, to draw an ordinate from P to any other diameter, since we have only to take V′M′ = T′V′, and join PM′.



213. The equation of the polar of any point $x'y'$ must be similar in form to that of the tangent (Art. 139), and is, therefore,

$$2y'y = p\,(x + x');$$

or we may prove it as follows, as in Art. 164:

If the point of contact of a tangent passing through $(x'y')$ be $(x''y'')$, then $(x'y')$ must satisfy the equation

$$2y''y = p\,(x + x''),$$

and we have

$$2y'y'' = p\,(x' + x'').$$

Hence $(x''y'')$ lies on the right line

$$2y'y = p\,(x + x').$$

So also must the point of contact of the other tangent through $(x'y')$; this line is, therefore, the line joining these points of contact. If we seek the point where this polar meets the axis of x, we get

$$x = -x'.$$

Hence we derive a theorem, which will be useful hereafter, *that the intercept which the polars of any two points cut off on the axis is equal to the intercept between perpendiculars from those points on that axis;* each of these quantities being equal to $(x' - x'')$.

DIAMETERS.

214. We have said, that if we take for axes any diameter and the tangent at its extremity, the equation will be of the form

$$y^2 = p'x.$$

We shall prove this again by actual transformation of the equation referred to rectangular axes ($y^2 = px$), because it is desirable to express the new p' in terms of the old p.

If we transform the equation $y^2 = px$ to parallel axes through any point $(x'y')$ on the curve, writing $x + x'$ and $y + y'$ for x and y, the equation becomes $y^2 + 2yy' = px.$

Now if, preserving our axis of x, we take a new axis of y, inclined to x at an angle $= \theta$, then our old y PN $=$ PM′ $\sin\theta$, and our old $x =$ V′M′ + PM′ $\cos\theta$. (See figure on last page.)

We, therefore, substitute $y \sin\theta$ for y, and $x + y\cos\theta$ for x, and our equation becomes

$$y^2 \sin^2\theta + 2y'y \sin\theta = px + py\cos\theta.$$

In order that this should reduce to the form $y^2 = px$, we must have

$$2y' \sin\theta = p\cos\theta, \quad \text{or } \tan\theta = \frac{p}{2y'}.$$

Now this is the very angle which the tangent makes with the angle of x, as we see from the equation

$$2y'y = p\,(x + x').$$

The equation, therefore, referred to a diameter and tangent, will take the form

$$y^2 = \frac{p}{\sin^2\theta}x, \text{ or } y^2 = p'x.$$

The quantity p' is called the parameter corresponding to the diameter V′M′, and we see that *the parameter of any diameter is to the principal parameter* (p), *inversely as the square of the sine of the angle which its ordinates make with the axis*, since $p' = \frac{p}{\sin^2\theta}$.

We can express the parameter of any diameter in terms of the co-ordinates of its vertex from the equation $\tan\theta = \frac{p}{2y'}$; hence,

$$\sin\theta = \frac{p}{\sqrt{(p^2 + 4y'^2)}} = \sqrt{\left(\frac{p}{p + 4x'}\right)};$$

hence $$p' = p + 4x.$$

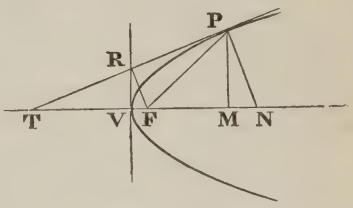
THE NORMAL.

215. The equation of a line through $(x'y')$ perpendicular to the tangent $2yy' = p(x + x')$ is

$$p(y - y') + 2y'(x - x').$$

If we seek the intercept on the axis of x we have

$$x\,(= \mathrm{VN}) = x' + \frac{p}{2};$$



and, since $\mathrm{VM} = x$, we must have

$$\mathrm{MN}\ (\text{the } \textit{subnormal}, \text{ Art. } 182) = \frac{p}{2}.$$

Hence *in the parabola the subnormal is constant, and equal to the semiparameter.* The normal itself

$$= \sqrt{(\mathrm{PM}^2 + \mathrm{MN}^2)} = \sqrt{\left(y'^2 + \frac{p^2}{4}\right)} = \sqrt{\left\{p\left(x' + \frac{p}{4}\right)\right\}}.$$

THE FOCUS.

216. A point situated on the axis of a parabola, at a distance from the vertex equal to one-fourth of the principal parameter, is called the *focus* of the curve.

This is the point which Art. 211 has led us to expect to find analogous to the focus of an ellipse; and we shall show, in the present section, that a parabola may in every respect be considered as an ellipse, having one of its foci at this distance, and the other at infinity.

To find the distance of any point on the curve from the focus.

The co-ordinates of the focus being $\left(\frac{p}{4}, 0\right)$, the square of its distance from any point is

$$\left(x' - \frac{p}{4}\right)^2 + y'^2 = x'^2 - \frac{px'}{2} + \frac{p^2}{16} + px' = \left(x' + \frac{p}{4}\right)^2.$$

Hence the distance of any point from the focus $= x' + \frac{p}{4}$.

This enables us to express more simply the result of Art. 214, and to say that *the parameter of any diameter is four times the distance of its extremity from the focus.*

217. The polar of the focus of a parabola is called the *directrix*, as in the ellipse and hyperbola.

Since the distance of the focus from the vertex $= \frac{p}{4}$, its polar is (Art. 213) a line perpendicular to the axis at the same distance on the other side of the vertex. The distance of any point from the directrix must, therefore, $= x' + \frac{p}{4}$.

Hence, by the last Article, *the distance of any point on the curve from the directrix is equal to its distance from the focus.*

We saw (Art. 191) that in the ellipse and hyperbola, the distance from the focus is to the distance from the directrix in the constant ratio e to 1. We see, now, that this is true for the parabola also, since in the parabola $e = 1$ (Art. 211).

The method given for mechanically describing an hyperbola, Art. 206, can be adapted to the mechanical description of the parabola, by simply making the angle ABR a right angle.

218. *The point where any tangent cuts the axis, and its point of contact, are equally distant from the focus.*

For, the distance from the vertex of the point where the tangent cuts the axis $= x'$ (Art. 212), its distance from the focus is, therefore, $x' + \frac{p}{4}$. Q. E. D.

219. *Any tangent makes equal angles with the axis and with the focal radius vector.*

This is evident from inspection of the isosceles triangle, which, in the last Article, we proved was formed by the axis, the focal radius vector, and the tangent.

This is only an extension of the property of the ellipse (Art. 187), that the angle TPF = T'PF'; for, if we suppose the focus F' to go off to infinity, the line PF' will become parallel to the axis, and TPF = PTF. (See figure at foot of p. 178).

Hence the tangent at the extremity of the focal ordinate cuts the axis at an angle of 45°.

220. *To find the length of the perpendicular from the focus on the tangent.*

It is evident (see figure, p. 182) that FR = FT sin FTR; but we have proved, Art. 214, that $\tan \mathrm{FTR} = \frac{p}{2y'}$, and, therefore,

$$\sin \mathrm{FTR} = \sqrt{\left(\frac{\frac{p}{4}}{x' + \frac{p}{4}}\right)}.$$

Hence

$$\mathrm{FR} = \sqrt{\left\{\frac{p}{4}\left(x' + \frac{p}{4}\right)\right\}},$$

or FR is a mean proportional between FV and FP.

It appears, also, from this expression, and from Art. 215, that FR is half the normal, as we might have inferred geometrically from the fact that TF = FN.

We might have obtained the same value of FR directly from the equation of the tangent, $2y'y = p\,(x + x')$, by Art. 31, and then, by reversing the process at the commencement of this Article, arrive at a more strictly algebraical proof of Art. 219.

221. To express the perpendicular from the focus in terms of the angles which it makes with the axis.

This is easy from Art. 220, for we have there given

$$\mathrm{FR} = \sqrt{\left\{\frac{p}{4}\left(x + \frac{p}{4}\right)\right\}},$$

and

$$\cos \alpha = \sin \mathrm{FTR} = \sqrt{\left(\frac{\frac{p}{4}}{x + \frac{p}{4}}\right)};$$

therefore,

$$\mathrm{FR} = \frac{p}{4 \cos \alpha}.$$

Hence *the locus of the extremity of the perpendicular from the focus on the tangent is a right line.*

For, taking the focus for pole, we have at once the polar equation

$$\rho \cos \alpha = \frac{p}{4}.$$

The equation shows that the right line is the tangent at the vertex.

Conversely, if from any point F we draw FR a radius vector to a right line VR, and draw PR perpendicular to it, the line PR will always touch a parabola having F for its focus.

We shall show hereafter how to solve generally questions of this class, where one condition less than is sufficient to determine a line is given, and it is required to find its *envelope*, that is to say, the curve which it always touches.

We leave, as a useful exercise to the reader, the investigation of the locus of the foot of the perpendicular by ordinary rectangular co-ordinates.

The equation of the tangent, the focus being the origin, can, therefore, be expressed

$$x \cos \alpha + y \sin \alpha + \frac{p}{4 \cos \alpha} = 0,$$

and hence we can express the perpendicular from any other point in terms of the angle it makes.

222. *To find the locus of the intersection of tangents which cut at right angles to each other.*

This can be solved from the form of the equation of the tangent given in the last Article,

$$x \cos^2 \alpha + y \sin \alpha \cos \alpha + \frac{p}{4} = 0.$$

The equation of a tangent perpendicular to this, that is, whose perpendicular makes an angle $= 90° + \alpha$ with the axis, is found by substituting $\cos \alpha$ for $\sin \alpha$, and $- \sin \alpha$ for $\cos \alpha$, or

$$x \sin^2 \alpha - y \sin \alpha \cos \alpha + \frac{p}{4} = 0.$$

α is eliminated by simply adding the equations, and we get

$$x + \frac{p}{2} = 0,$$

the equation of *the directrix*, since the distance of focus from directrix $= \frac{p}{2}$.

223. The theorem of the last Article is only a particular case of another more general one, which follows at once geometrically from the principles already laid down.

The angle between any two tangents is half the angle between the focal radii vectores to their points of contact.

For, from the isosceles PFT, the angle PTF which the tangent makes with the axis is half the angle PFN, which the focal radius makes with it. Now, the angle between any two tangents is equal to the difference of the angles they make with the axis, and the angle between two focal radii is equal to the difference of the angles which *they* make with the axis.

We can now obtain the theorem of the last Article, for if two tangents make with each other an angle of 90°, the focal radii must make with each other an angle of 180°, therefore, the two tangents must be drawn at the extremities of a chord through the focus, and, therefore, from the definition of the directrix, must meet on the directrix.

224. It can be proved, by a method precisely similar to that used in Art. 192, that the line joining the pole of any focal chord to the focus is perpendicular to the chord. Leaving this method to the student's own examination, we shall prove another general proposition under which this is included, and which we shall show, in Chap. XII., to be true for all the conic sections: *the line joining the focus to the intersection of two tangents bisects the angle which their points of contact subtend at the focus.*

Let us use the form of the equation of the tangent given in the last Article,

$$x\cos^2\alpha + y\sin\alpha\cos\alpha + \frac{p}{4} = 0,$$

and

$$x\cos^2\beta + y\sin\beta\cos\beta + \frac{p}{4} = 0,$$

where α and β are the angles made by the perpendiculars on these tangents with the axis; and, subtracting, we get, for the line joining the intersection to the focus,

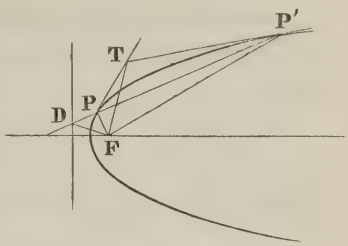
$$x\sin(\alpha+\beta) - y\cos(\alpha+\beta) = 0.$$

This is the equation of a line making the angle $\alpha + \beta$ with the axis of x. Now, VFP = 2α and VFP′ = 2β, therefore, the line making an angle with the axis = $\alpha + \beta$ must bisect the angle PFP′.

Cor. 1. If we take the case where the angle PFP′ = 180°, then

the tangents TP, TP′ will intersect on the directrix, and the angle TFP = 90°.

Cor. 2. If any chord PP′ cut the directrix in D, then FD is the external bisector of the angle PFP′, for we have proved FT to be the internal bisector, but D is the pole of FT (since it is the intersection of PP′, the polar of T, with the directrix, the polar of F); therefore, DF is perpendicular to FT (Cor. 1), and, therefore, the external bisector.

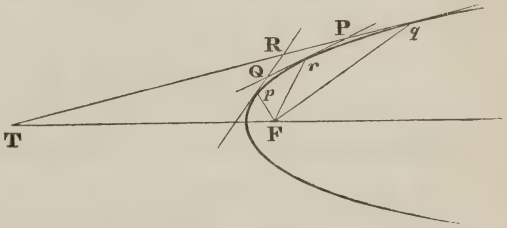


Cor. 3. If any variable tangent to the parabola meet two fixed tangents, the angle subtended at the focus by the portion of the variable tangent intercepted between the fixed tangents, is the supplement of the angle between the fixed tangents.

For, it was proved (Art. 223), that the angle QRT was half $p\mathrm{F}q$, but, by the present Article, $p\mathrm{FQ} = r\mathrm{FQ}$ and $q\mathrm{FP} = r\mathrm{FP}$, therefore PFQ is also half $p\mathrm{F}q$, and, therefore,

PFQ is = QRT,

or is the supplement of PRQ.

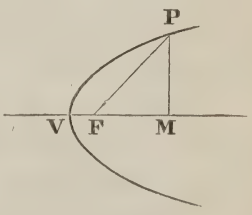


Cor. 4. From the last corollary it follows, that a circle described through PQR must pass through F, since the angle contained in the segment PQF will be the supplement of that contained in PQR. That is to say: *The circle circumscribing the triangle formed by any three tangents to a parabola will pass through the focus.*

225. *To find the polar equation of the parabola, the focus being the pole.*

We proved (Art. 216) that the focal radius

$$= x' + \frac{p}{4} = \mathrm{VM} + \frac{p}{4} = \mathrm{FM} + \frac{p}{2} = \rho\cos\theta + \frac{p}{2}.$$



Hence

$$\rho = \frac{\frac{p}{2}}{1 - \cos\theta}.$$

This is exactly what the equation of Art. 193 becomes, if we suppose $e = 1$ (Art. 211).

In this equation θ is supposed to be measured from the side FM; if we suppose it measured from the side FV, the equation becomes

$$\rho = \frac{\frac{p}{2}}{1 + \cos\theta}.$$

This equation may be written

$$\rho \cos^2 \tfrac{1}{2}\theta = \frac{p}{4};$$

or

$$\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta = \left(\frac{p}{4}\right)^{\frac{1}{2}},$$

and is, therefore, one of a class of equations,

$$\rho^n \cos n\theta = a^n,$$

some of whose properties we shall mention hereafter.

CHAPTER XII.

EXAMPLES AND MISCELLANEOUS PROPERTIES OF THE CONIC SECTIONS.

226. The method of applying algebra to problems relating to conic sections is essentially the same as that employed in the case of the right line and circle, and will present no difficulty to any reader who has carefully worked out the examples given in Chaps. III. and VII. We, therefore, only think it necessary to select a few out of the great multitude of examples which lead to loci of the second order, and we shall then add some properties of conic sections which it was not found convenient to insert in the preceding chapters.

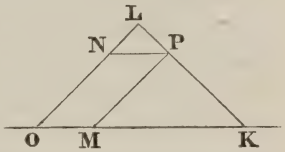
Example 1. *To find the locus of a point such that its distance from a fixed point* $(x'y')$ *may be in a constant ratio to its distance from a fixed right line.*

If we take the fixed line for the axis of x, the equation of the locus will be $(x - x')^2 + (y - y')^2 = e^2 y^2$.

Since the term xy is wanting in this equation, it will represent an ellipse, hyperbola, or parabola, according as the co-efficient of y^2 is positive, negative, or cypher (Art. 129), that is, according as e is less, greater, or equal to 1.

This is the converse of Art. 191.

Ex. 2. *A line of constant length moves about in the legs of a given angle: to find the locus described by a fixed point on it.*



By similar triangles

$$\text{OL} = \frac{ly}{m}, \text{ and } \text{OK} = \frac{lx}{n},$$

(denoting PL by n, PK by m, and LK by l),

but since $\quad \text{LK}^2 = \text{OL}^2 + \text{OK}^2 - 2\text{OK.OL}\cos\omega,$

we have $$l^2 = \frac{l^2y^2}{m^2} + \frac{l^2x^2}{n^2} - \frac{2l^2xy\cos\omega}{mn},$$

or $$\frac{x^2}{n^2} + \frac{y^2}{m^2} - \frac{2xy\cos\omega}{mn} = 1;$$

the equation of an *ellipse* having the point O for its centre, since $\text{B}^2 - 4\text{AC}$ is here negative, being $= -\frac{4}{m^2n^2}\sin^2\theta$.

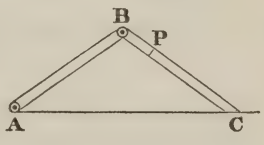
This is an extension of the principle of the elliptic compasses (Art. 176).

Ex. 3. *If* P *be a fixed point, and* LK *any right line drawn through it, to find the locus of intersection of the parallels to* OK, OL, *through the points* L *and* K.

Ex. 4. *Or of perpendiculars erected to* OK, OL, *through the same points.*

Ex. 5. *If a point* Q *be taken on* LK *so that* QK = PL, *to find its locus.*

Ex. 6. *Two equal rulers,* AB, AC, *are connected by a pivot at* B; *the extremity* A *is fixed, while the extremity* C *is made to traverse the right line* AC; *find the locus described by any fixed point* P *on* BC.



Ex. 7. *Given base and difference of base angles of a triangle: to find the locus of vertex.*

We may proceed exactly as at page 87, where the *sum* of the

base angles is given. The locus will be found to be an equilateral hyperbola, of which the base is a diameter. The difference of base angles being given, it is easy to see that the internal and external bisectors of the vertical angle must be parallel to fixed lines, and these lines will be parallel to the asymptotes of the locus. Conversely if we consider the triangle whose base is any diameter of an equilateral hyperbola, and whose vertex is on the curve, the sides are parallel to conjugate diameters (Art. 174); but conjugate diameters of an equilateral hyperbola make equal angles with the asymptotes (Art. 169).

Ex. 8. *Given base and the product of the tangents of the base angles of a triangle : find the locus of vertex.*

It will be a conic section, of which the extremities of the base are vertices. This is the converse of Art. 166.

Ex. 9. *Given base and the product of the tangents of the halves of the base angles : find the locus of vertex.*

It will be an ellipse, of which the extremities of the base are the foci.

Ex. 10. *Given base and sum of sides of a triangle : find the locus of the centre of the inscribed circle.*

It may be immediately inferred, from the last two examples, that the locus is an ellipse, whose vertices are the extremities of the given base.

Ex. 11. *Given the vertical angle of a triangle in magnitude and position, and also the area : find the locus of a point dividing the base in a given ratio.*

227. *If a curve be such that the distance of any point of it from a fixed point can be expressed as a rational function of the first degree of its co-ordinates, then the curve must be a conic section, and the fixed point its focus.**

For, if the distance can be expressed

$$\rho = Ax + By + C,$$

since $Ax + By + C$ is proportional to the perpendicular let fall on the right line whose equation is $(Ax + By + C = 0)$, the equation signifies that the distance of any point of the curve from the fixed

* See *O'Brien's Co-ordinate Geometry*, p. 85.

point is in a constant ratio to its distance from this line, and, therefore (Art. 226, Ex. 1), the curve is a conic section, of which the fixed point is a focus.

If ρ be a rational function of the co-ordinates of a degree *higher than the first*, we have

$$\sqrt{(x^2 + y^2)} = A + Bx + Cy + Dx^2 + Exy + \&c.,$$

and it is evident that, if we clear the equation of radicals, the curve will be of a degree higher than the second.

228. Ex. 1. *If any variable tangent to a central conic section meet two fixed parallel tangents, it will intercept portions on them, whose rectangle is constant, and equal to the square of the semi-diameter parallel to them.*

Let us take for axes the diameter parallel to the tangents and its conjugate, then the equation of the curve will be

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

and the equation of the variable tangent

$$\frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 1.$$

The intercepts on the fixed tangents are found by making x alternately $= \pm a'$, and we get

$$y = \frac{b'^2}{y'}\left(1 \mp \frac{x'}{a'}\right),$$

and, therefore, their product is

$$\frac{b'^4}{y'^2}\left(1 - \frac{x'^2}{a'^2}\right);$$

but, from the equation of the curve

$$\frac{y'^2}{b'^2} = \left(1 - \frac{x'^2}{a'^2}\right);$$

therefore the product $= b'^2$.

Ex. 2. The same construction remaining, *the rectangle under the segments of the variable tangent is equal to the square of the diameter parallel to* it.

For, the intercept on either of the parallel tangents is to the adjacent segment of the variable tangent as the parallel semi-

diameters (Art. 147); therefore, the rectangle under the intercepts of the fixed tangents is to the rectangle under the segments of the variable tangent as the *squares* of these semidiameters; and, since the first rectangle is equal to the square of the semidiameter parallel to it, the second rectangle must be equal to the square of the semidiameter parallel to *it*.

Ex. 3. *If any tangent meet any two conjugate diameters, the rectangle under its segments is equal to the square of the parallel semidiameter.*

Take for axes the semidiameter parallel to the tangent and its conjugate, then the equation of the curve will be

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

and, if the equation of any diameter be

$$y = \frac{y'}{x'} x,$$

that of its conjugate will be

$$\frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 0 \text{ (Art. 166)}.$$

The intercepts on the tangent are found by making $x = a'$ to be

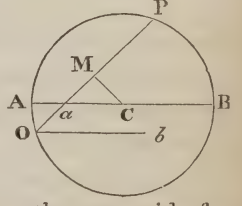
$$y = \frac{y'}{x'} a', \text{ and } y = \frac{b'^2}{a'} \frac{x'}{y'},$$

whose rectangle is evidently $= b'^2$.

We might, in like manner, have given a purely algebraical proof of Ex. 2.

Hence, also, *if the points be joined to the centre where two parallel tangents meet any tangent, the joining lines will be conjugate diameters.*

Ex. 4. Given, in magnitude and position, two conjugate diameters, Oa, Ob, of a central conic, to determine the axes.



The following construction is founded on the theorem proved in the last example: Through a, the extremity of either diameter, draw a parallel to the other, it must of course be a tangent to the curve. Now on Oa take a point P, such that the rectangle Oa.aP = Ob^2 (on the side remote from O for the ellipse, on the same side for

the hyperbola), and describe a circle through O, P, having its centre on ac, then the lines OA, OB, are the axes of the curve; for, since the rectangle $Aa \,.\, aB = Oa \,.\, aP = Ob^2$, the lines OA, OB are conjugate diameters, and since AB is a diameter of the circle, the angle AOB is right.

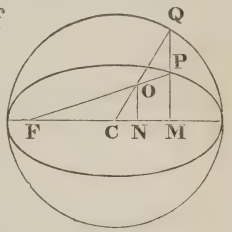
In the case of the hyperbola it is more simple to find the asymptotes by Art. 198, and thence the axes.

Ex. 5. *Given any two semidiameters, if from the extremity of each an ordinate be drawn to the other, the triangles so formed will be equal in area.*

Ex. 6. *Or if tangents be drawn at the extremity of each, the triangles so formed will be equal in area.*

229. We shall in this Article illustrate the use of the *eccentric angle* in simplifying formulæ.

Ex. 1. *To find the locus of the intersection of the focal radius vector* FP *with the radius of the circle* CQ.



Let the central co-ordinates of P be $x'y'$, of O xy, then we have, from the similar triangles, FON, FPM,

$$\frac{y}{x+c} = \frac{y'}{x'+c} = \frac{b\sin\phi}{a(e+\cos\phi)}$$

(if we substitute for x' and y' the values $a\cos\phi$, $b\sin\phi$, Art. 175). Now, since ϕ is the angle made with the axis by the radius vector to the point O, we at once obtain the polar equation of the locus by writing $\rho\cos\phi$ for x, $\rho\sin\phi$ for y, and we find

$$\frac{\rho}{c+\rho\cos\phi} = \frac{b}{a(e+\cos\phi)},$$

or

$$\rho = \frac{bc}{c+(a-b)\cos\phi}.$$

Hence (Art. 193) the locus is an ellipse, of which C is one focus, and it can easily be proved that F is the other.

Ex. 2. *The normal at* P *is produced to meet* CQ: *find the locus of their intersection.*

The equation of the normal is (Art. 181)

$$\frac{a^2x}{x'} - \frac{b^2y}{y'} = c^2,$$

or (Art. 175) $$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = c^2;$$

but we may, as in the last example, write $\rho\cos\phi$ and $\rho\sin\phi$ for x and y, and the equation becomes

$$(a - b)\rho = c^2,$$

or $$\rho = a + b.$$

The locus is, therefore, a circle concentric with the ellipse.

Ex. 3. It is useful in astronomy to express the angle PFC in terms of the angle ϕ. It will be found that

$$\tan\tfrac{1}{2}\mathrm{PFC} = \sqrt{\left(\frac{1-e}{1+e}\right)}\tan\tfrac{1}{2}\phi.$$

Ex. 4. *If from the vertex of an ellipse a radius vector be drawn to any point on the curve, find the locus of the point where a parallel radius through the centre meets the tangent at the point.*

The tangent of the angle made with the axis by the radius vector to the vertex $= \frac{y'}{x' + a}$, therefore, the equation of the parallel radius through the centre is

$$\frac{y}{x} = \frac{y'}{x' + a} = \frac{b\sin\phi}{a(1 + \cos\phi)},$$

and we seek the locus of the intersection of this line with the tangent

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1 \text{ (Art. 178)}.$$

The first equation may be written

$$\frac{y}{x} = \frac{b}{a}\frac{(1 - \cos\phi)}{\sin\phi},$$

or $$\frac{y}{b}\sin\phi + \frac{x}{a}\cos\phi = \frac{x}{a}.$$

The locus is, therefore, $\frac{x}{a} = 1$, the tangent at the other extremity of the axis.

The same investigation will apply, if the first radius vector be drawn through any point of the curve, by substituting a' and b' for a and b, the locus will then be the tangent at the diametrically opposite point.

230. Ex. 1. *To find the locus of the intersection of tangents to an ellipse or hyperbola which cut each other at right angles.*

Let us denote by p, p', the perpendiculars from the centre on the two tangents, then (Art. 173)

$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha;$$

but, since the tangents are at right angles, we have

$$p'^2 = a^2\sin^2\alpha + b^2\cos^2\alpha.$$

Now the square of the distance of the centre from the intersection of any two lines at right angles, is equal to the sum of the squares of its distances from the lines themselves, or

$$\rho^2 = p^2 + p'^2 = a^2 + b^2.$$

The locus is, therefore, a circle concentric with the given conic.

Ex. 2. *To draw a normal to an ellipse or hyperbola passing through a given point.*

Let the point on the curve at which the normal is drawn be XY, then its equation is (Art. 181)

$$\frac{a^2x}{X} - \frac{b^2y}{Y} = c^2;$$

and if this normal passes through a fixed point $x'y'$, we have the relation

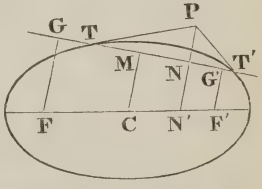
$$\frac{a^2x'}{X} - \frac{b^2y'}{Y} = c^2.$$

Hence the points on the curve, whose normals will pass through $(x'y')$, are the points of intersection of the given curve with the hyperbola

$$c^2xy = a^2x'y - b^2y'x.$$

231. The equation of the polar of any point being given by the equation

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$



the student will have no difficulty in finding the lengths of the perpendiculars on this polar from the centre and the foci, or in establishing the following theorems (see note, p. 168).

Ex. 1. $CM \cdot PN' = b^2$. This is analogous to the theorem that the rectangle under the normal and the central perpendicular is constant.

Ex. 2. $PN' . NN' = \frac{b^2}{a^2}(a^2 - e^2x'^2)$. When P is on the curve this equation gives us the known expression for the normal $= \frac{bb'}{a}$ (Art. 182).

Ex. 3. $FG . F'G' = CM . NN'$. When P is on the curve this theorem becomes $FG . F'G' = b^2$.

232. *If two fixed points* $(x'y')$, $(x''y'')$, *on an hyperbola, be joined to any variable point* $(x'''y''')$, *the portion which they intercept on either asymptote is constant.*

If we take the asymptotes for our axes of co-ordinates, the equation of any chord is (Art. 203)

$$x'''y + y'x = y'x''' + k^2.$$

If we make in this equation $y = 0$, we find, for the intercept from the origin on the axis of x,

$$x = x' + x'''.$$

In like manner, the intercept made by the other chord is

$$x = x'' + x''',$$

and the difference between these $x' - x''$ is independent of the position of the variable point $x'''y'''$.

233. Ex. 1. *To find the angle subtended at the focus by the tangent drawn to a central conic from any point* (xy).

Let the point of contact be $(x'y')$, the centre being the origin, then, if the focal radii to the points (xy), $(x'y')$, be ρ, ρ', and make angles θ, θ', with the axis, it is evident that

$$\cos\theta = \frac{x + c}{\rho}, \quad \sin\theta = \frac{y}{\rho}; \quad \cos\theta' = \frac{x' + c}{\rho'}, \quad \sin\theta' = \frac{y'}{\rho'}.$$

Hence

$$\cos(\theta - \theta') = \frac{(x + c)(x' + c) + yy'}{\rho\rho'};$$

but from the equation of the tangent we must have

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

Substituting this value of yy', we get

$$\rho\rho' \cos(\theta - \theta') = xx' + cx + cx' + c^2 - \frac{b^2}{a^2}xx' + b^2,$$

or $\rho\rho' \cos(\theta - \theta') = e^2xx' + cx + cx' + c^2 = (a + ex)(a + ex')$;

or, since $\rho' = a + ex'$, we have

$$\cos(\theta - \theta') = \frac{a + ex}{\rho}.*$$

Since this value depends solely on the co-ordinates (xy), and does not involve the co-ordinates of the point of contact, either tangent drawn from (xy) subtends the same angle at the focus (see Art. 224).

Ex. 2. It can in like manner be proved, that if the conic section be a parabola,

$$\cos(\theta - \theta') = \frac{x + \frac{p}{4}}{\rho}.$$

Ex. 3. *If normals be drawn at the extremities of any focal chord, a line drawn through their intersection parallel to the axis will bisect the chord.*

Ex. 4. *A line joining the intersection of the focal normals to the intersection of the corresponding focal tangents will pass through the other focus.*

Ex. 5. *If the ordinate at any point of a conic be produced to meet the tangent at a focus, its whole length will be equal to the distance of the point from that focus.*

Ex. 6. *If from the focus a line be drawn, making a given angle with any tangent, find the locus of the point where it meets it.*

234. *If through any point on a conic two lines at right angles to each other be drawn to meet the curve, the line joining their extremities will pass through a fixed point on the normal.*

Let us take for axes the tangent and normal at the given point, then the equation of the curve must be of the form

$$Ax^2 + Bxy + Cy^2 + Ey = 0$$

(for F = 0, because the origin is on the curve, and D = 0 (Art. 132), because the tangent is supposed to be the axis of x, whose equation is $y = 0$).

Now let the equation of *any* two lines through the origin be

$$x^2 + pxy + qy^2 = 0.$$

* This very simple expression is taken from *O'Brien's Co-ordinate Geometry*, p. 156.

Multiply this equation by A, and subtract it from that of the curve, and we get

$$(B - Ap)\,xy + (C - Aq)\,y^2 + Ey = 0.$$

This (Art. 48) is the equation of a figure passing through the points of intersection of the lines and conic; but it may evidently be resolved into the two equations

$$y = 0,$$

the tangent at the given point, and

$$(B - Ap)\,x + (C - Aq)\,y + E = 0,$$

which must be the equation of the chord joining the extremities of the given lines.

The point where this chord meets the normal (the axis of y) is $y = \dfrac{E}{Aq - C}$; but if the lines are at right angles $q = -1$ (Art. 74), and the intercept on the normal has the constant length

$$= \frac{-E}{A + C}.$$

This theorem will be equally true if the lines be drawn so as to make with the normal angles, the product of whose tangents is constant, for, in this case, q is constant, and, therefore, the intercept $\dfrac{E}{Aq - C}$ is constant.

235. *Given two conic sections, to find the locus of the pole, with respect to one, of any tangent to the other.*

Let their equations be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

the polar of any point, with regard to the second, is (Art. 139)

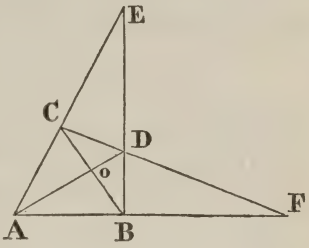
$$(2Ax' + By' + D)\,x + (2Cy' + Bx' + E)\,y + Dx' + Ey' + 2F = 0.$$

But the condition that this should touch the first is (Art. 163)

$$a^2.(2Ax' + By' + D)^2 + b^2.(2Cy' + Bx' + E)^2 = (Dx' + Ey' + 2F)^2.$$

This condition, which must be satisfied by the point $(x'y')$, is the equation of its locus, and is plainly of the second degree.

236. Ex. 1. *If a quadrilateral,* ABCD, *be inscribed in a conic section, any of the points* E, F, O, *is the pole of the line joining the other two.*



This is only another mode of stating the first theorem in Art. 143; for since EC, ED, are two lines drawn through the point E, and CD, AB, one pair of lines joining the points where they meet the conic, these lines must intersect on the polar of E; so must also AD and CB; therefore, the line OF is the polar of E. In like manner it can be proved that EF is the polar of O, and EO the polar of F.

The theorem in Art. 143 also furnishes us with a solution of the problem, "To draw a tangent to a given conic section from a point outside, with the help of the ruler only."

For we have only to draw any two lines through the given point E, and to complete the quadrilateral as in the figure, then the line OF will meet the conic in two points, which, being joined to E, will give the two tangents required.

Ex. 2. *If a quadrilateral be* circumscribed *about a conic section, any diagonal is the polar of the intersection of the other two.*

We shall prove this theorem, as we might have proved the last, by means of the harmonic properties of a quadrilateral. It was proved (Art. 59) that EA, EO, EB, EF, are an harmonic pencil. Hence, since EA, EB, are, by hypothesis, two tangents to a conic section, and EF a line through their point of intersection, by Art. 144 EO must pass through the pole of EF; for the same reason, FO must pass through the pole of EF: this pole must, therefore, be O.

Ex. 3. *Given four points on a conic section, to find the locus of its centre.*

Let us take for axes any two opposite sides of the quadrilateral passing through the four given points. Now, the general equation of a conic being

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + 1 = 0,$$

the intercepts on the axis of x are found from the equation

$$Ax^2 + Dx + 1 = 0;$$

and these intercepts being given, A and D are given. In like manner, it is proved that C and E are given, so that B is the only indeterminate which remains in the equation. Now the co-ordinates of the centre must satisfy the equations

$$2Ax + By + D = 0,$$
$$2Cy + Bx + E = 0;$$

and if from these equations we eliminate B, we obtain the equation of the locus,

$$2Ax^2 - 2Cy^2 + Dx - Ey = 0.$$

The condition that the axes should touch the curve is (Art. 129) $D^2 = 4AF$ and $E^2 = 4CF$. It will be found that, in this case, the locus will become the line bisecting the chord of contact, as is evident geometrically.

Ex. 4. *Given four points on a conic section, the polar of any fixed point will pass through a fixed point.*

The equation of the polar of any point $(x'y')$ is

$$(2Ax' + By' + D)\,x + (2Cy' + Bx' + E)\,y + Dx' + Ey' + 2F = 0.$$

Now, if in this equation we suppose B indeterminate, since it only enters in the first degree, the line must pass through a fixed point (Art. 69).

Ex. 5. *Given four tangents to a conic section, the locus of the centres is the line joining the middle points of the diagonals.*

Ex. 6. *And the locus of the pole of any fixed right line is a right line.*

237. *To find the locus of the pole of a fixed line with regard to a series of concentric and confocal conic sections.*

We know that the pole of any line $\left(\frac{x}{m} + \frac{y}{n} = 1\right)$, with regard to the conic $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$, is found from the equations $mx = a^2$ and $ny = b^2$ (Art. 164).

Now, if the foci of the conic are given, $a^2 - b^2 = c^2$ is given; hence, the locus of the pole of the fixed line is

$$mx - ny = c^2,$$

the equation of a right line perpendicular to the given line.

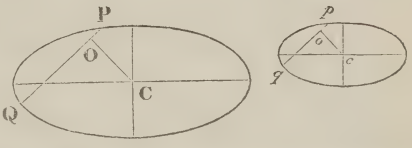
If the given line touch one of the conics, its pole will be the point of contact (Art. 132). Hence, *given two confocal conics, if*

we draw any tangent to one and tangents to the second where this line meets it, these tangents will intersect on the normal to the first conic.

We have already mentioned two other important properties of confocal conics (Arts. 187 and 189).

SIMILAR CONIC SECTIONS.

238. Any two figures are said to be *similar and similarly placed,* if radii vectores drawn to the first from a certain point O are in a constant ratio to parallel radii drawn to the second from another point *o*. If it be possible to find any two such points O and *o*, we can find an infinity of others; for, take any point C, draw *oc* parallel to OC, and in the constant ratio $\frac{op}{OP}$, then from the similar triangles OCP, *ocp*, *cp* is parallel to CP, and in the given ratio. In like manner, any other radius vector through *c* can be proved to be proportional to the parallel radius through C.



If two *central conic sections* be similar, all diameters of the one are constantly proportional to the parallel diameters of the other, since the rectangles OP.OQ, *op*.*oq*, are proportional to the squares of the parallel diameters (Art. 147).

239. We now propose to investigate the condition that two conic sections, whose equations are given,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$
$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

should be similar, and similarly placed.

The equation of the first, referred to its centre as origin, must (Art. 149) be of the form

$$Ax^2 + Bxy + Cy^2 = F',$$

and the square of any semi-diameter

$$R^2 = \frac{F'}{A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta};$$

the square of a parallel semi-diameter of the second is

$$\rho^2 = \frac{f}{a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta}.$$

The ratio $\frac{R^2}{\rho^2}$ cannot be independent of θ unless we have

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c}.$$

Hence, *two conic sections will be similar, and similarly placed, if the co-efficients of the highest powers of the variables are the same in both, or only differ by a constant multiplier.*

240. It is evident that the directions of the axes of similar conics must be the same, since the greatest and least diameters of one must be parallel to the greatest and least diameters of the other.

If the diameter of one become infinite, so must also the parallel diameter of the other, that is to say, *the asymptotes of similar and similarly placed hyperbolæ are parallel.* The same thing follows from the result of the last Article, since (Art. 152) the directions of the asymptotes are wholly determined by the highest terms of the equation.

Similar conics have the same eccentricity; for $\frac{a^2 - b^2}{a^2}$ must be $= \frac{m^2a^2 - m^2b^2}{m^2a^2}$. Similar and similarly placed conic sections have hence sometimes been defined as those whose axes are parallel, and which have the same eccentricity.

If two hyperbolæ have parallel asymptotes they are similar, for their axes must be parallel, since they bisect the angles between the asymptotes (Art. 152), and the eccentricity wholly depends on the angle between the asymptotes (Art. 161).

241. Since the eccentricity of all parabolæ is constantly = 1, we should be led to infer that all parabolæ are similar and similarly placed, the direction of whose axes is the same. Since, however, the investigation of Art. 239 only applies to central conics, it is necessary to give a separate proof of this theorem. The equation of one parabola, referred to its vertex, being $y^2 = px$, or

$$\rho = \frac{p\cos\theta}{\sin^2\theta},$$

it is plain that a parallel radius vector through the vertex of the other will be to this radius in the constant ratio $\frac{p}{p'}$.

242. Ex. 1. *If on any radius vector to a conic section through a fixed point* O, OQ *be taken in a constant ratio to* OP, *find the locus of* Q.

The investigation is precisely the same as at page 98, and the locus will be found to be a conic similar to the given conic, and similarly placed.

The point O may be called *the centre of similitude* of the two conics, and it can be proved, as at page 99, that the centre of similitude is the point where common tangents intersect.

Ex. 2. *If a pair of radii be drawn through a centre of similitude of two similar conics, the chords joining their extremities will be either parallel, or will meet on the chord of intersection of the conics.*

This is proved precisely as at page 112.

Ex. 3. *Given three similar conics, their six centres of similitude will lie three by three on right lines* (see figure, page 114).

Ex. 4. *If any line cut two similar and concentric conics, its parts intercepted between the conics will be equal.*

Any chord of the outer conic which touches the interior, will be bisected at the point of contact.

These are proved in the same manner as the theorems at page 172, which are but particular cases of them; for the asymptotes of any hyperbola may be considered as a conic section similar to it, since the highest terms in the equation of the asymptotes are the same as in the equation of the curve.

Ex. 5. *If a tangent drawn at* V, *the vertex of the inner of two concentric and similar ellipses, meet the outer in the points* T *and* T′, *then any chord of the inner drawn through* V *is half the algebraic sum of the parallel chords of the outer through* T *and* T′.

243. Two figures will be similar, although not similarly placed, if the proportional radii make a constant angle with each other, instead of being parallel; so that, if we could imagine one of the figures turned round through the given angle, they would be then both similar and similarly placed.

If we wish to find the condition that the two conic sections,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$
$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

should be similar, even though not similarly placed, we have only to transform the first equation to axes making any angle θ with the given axes, and examine whether any value can be assigned to θ which will make the co-efficients of the highest powers of the variables proportional to the co-efficients in the second equation. We thus obtain the two conditions (Art. 153)

$$\frac{A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta}{B(\cos^2\theta - \sin^2\theta) - 2(A - C)\sin\theta\cos\theta} = \frac{a}{b},$$
$$\frac{A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta}{B(\cos^2\theta - \sin^2\theta) - 2(A - C)\sin\theta\cos\theta} = \frac{c}{b},$$

from which, if we eliminate θ, we shall have the required condition.

Now if the curves be hyperbolæ we may evade the trouble of elimination, for, in order that it should be possible to turn the asymptotes of one parallel to those of the other, it is necessary that they should contain the same angle, or (Art. 74) that

$$\frac{B^2 - 4AC}{(A + C)^2} = \frac{b^2 - 4ac}{(a + c)^2}.$$

This, therefore, must be the result of the elimination if the curves be hyperbolæ, but it is evident that the result of the elimination must be the same whatever be the values of a, b, c; hence, in every case, the result of the elimination, as the reader may easily verify, will be

$$\frac{B^2 - 4AC}{(A + C)^2} = \frac{b^2 - 4ac}{(a + c)^2}.*$$

This is, therefore, in general, the condition that two conic sections should have the same eccentricity.

We wish the reader to examine attentively the force of the proof employed in this Article, because it affords one of the earliest examples of the use which we shall make of *the law of continuity*, viz., that, being given any figure in its most general state (where certain points, lines, &c., are real), then any property of that figure not involving those points or lines will be true of any particular state of that figure, even where some of those lines

* This condition was communicated to me by Mr. Jellett.

or points are imaginary. Thus, for example, any property of the hyperbola not involving the asymptotes will be true of the ellipse (whose asymptotes are imaginary); and we shall be safe in asserting this, even if it were the consideration of the asymptotes which had originally led us to the property of the hyperbola. We should, in most cases, be able to justify this inference by showing that there was some other method of proof which must hold good, and bring out the same result in all cases; and that, therefore, if we can ascertain in any case what that result must be, we know it for every case. In the present question, for instance, the method of transformation must bring out the same result in all cases; but in the case of the hyperbola, the consideration of the asymptotes showed us at once what this result must be: we were thus enabled, without the trouble of elimination, to arrive immediately at the required condition.

THE CONTACT OF CONIC SECTIONS.

244. We proved (Art. 18) that we obtain an equation of the mn^{th} degree to determine the co-ordinates of the points of intersection of two curves of the m^{th} and n^{th} degrees. Hence, *two conic sections will in general intersect in four points.*

If two of these points of intersection coincide, the conic sections are said to touch each other, and the line joining the coincident points will be the common tangent.

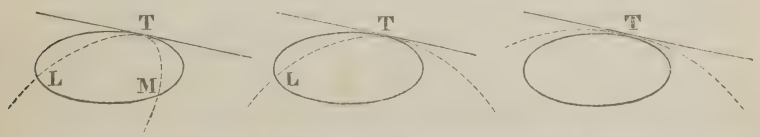
Let the equations of the conics, referred to the tangent and normal, be (Art. 234)

$$Ax^2 + Bxy + Cy^2 + Ey = 0,$$
$$A'x^2 + B'xy + C'y^2 + E'y = 0,$$

then the equation of the line (LM) joining the other two points of intersection will be, as in Art. 234,

$$(BA' - AB')\,x + (CA' - AC')\,y + (EA' - AE') = 0.$$

This is called *a contact of the first order.*



Now the contact of the conics will evidently be more close if

three of their points of intersection coincide. In this case one of the points L, M, must coincide with T, the line LM must pass through the origin, and we must have the condition

$$EA' - AE' = 0.$$

This is called *a contact of the second order*. Curves which have a contact higher than the first order are said to *osculate*, and it appears that conics which osculate, in general, meet each other in *one* other point.

The contact of two conics will be the closest possible when they have *four* consecutive points in common. In this case the line LM must coincide with the tangent at T ($y = 0$), and we must have the two conditions

$$EA' - AE' = 0, \qquad BA' - AB' = 0.$$

This is called *a contact of the third order;* and since two conic sections cannot have more than four points common, it is the highest order of contact which two different conics can have.

Hence, if the equation of one curve be

$$x^2 + Bxy + Cy^2 + Ey = 0,$$

that of the other will be

$$x^2 + Bxy + C'y^2 + Ey = 0.$$

245. Hence an infinity of conic sections can be drawn having a contact of the third order with a given conic at a given point, and any *one* condition will enable us to determine the touching conic. Thus, for example, the *parabola* having a contact of the third order with the conic

$$x^2 + Bxy + Cy^2 + Ey = 0$$

will be

$$x^2 + Bxy + \frac{B^2}{4}y^2 + Ey = 0.$$

We cannot describe a *circle* to have a contact of the third order with a given conic, because *two* conditions must be fulfilled in order that this equation should represent a circle; or, in other words, we cannot describe a circle through four consecutive points on a conic, since three points are sufficient to determine a circle. We can, however, easily find the equation of the circle passing through *three* consecutive points on the curve. This circle is called the *osculating circle*, or the *circle of curvature*.

The equation of the conic being

$$Ax^2 + Bxy + Cy^2 + Ey = 0,$$

that of any circle touching it is (Art. 82, IV.)

$$Ax^2 + Ay^2 - 2Axy = 0,$$

and the condition that the circle should osculate is (Art. 244)

$$E = 2Ar, \quad \text{or } r = \frac{E}{2A}.$$

The quantity r is called the *radius of curvature* of the conic at the point T.

246. *To find an expression for the radius of curvature at any point of an ellipse.*

It is plain, from the last Article, that this would be done if we could transform the equation to the tangent and normal at the point.

The equation referred to a diameter through the point and its conjugate $\left(\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1\right)$, when transferred to the given point, becomes

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = \frac{2x'}{a'}.$$

The axes are now a tangent and diameter through the point, but if, allowing the axis of y to remain unaltered, we make the normal the axis of x, we must substitute (Art. 13)

$$\frac{x}{\sin\theta} \text{ for } x, \text{ and } y - \frac{x}{\tan\theta} \text{ for } y,$$

where θ is the angle which the diameter makes with the tangent.

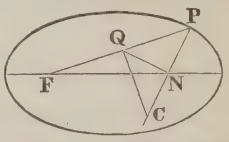
The new co-efficient of x will, therefore, be $\frac{2}{a' \sin\theta}$, and of y^2 will be $\frac{1}{b'^2}$; hence the radius of curvature will be $\frac{b'^2}{a' \sin\theta}$. Now $a' \sin\theta$ is the perpendicular from the centre on the tangent, therefore the radius of curvature $= \frac{b'^2}{p}$.

Hence, also (Art. 170), $$R = \frac{b'^3}{ab}.$$

247. This value enables us to construct simply for the radius of curvature at any point. We proved (Art. 182) that the length

of the normal $= \frac{bb'}{a}$, and that $\cos\psi = \frac{b}{b'}$ (ψ being the angle between the focal radius and the normal); hence

$$R = \frac{N}{\cos^2\psi}.$$



If, therefore, we erect a perpendicular to the normal at the point where it meets the axis, and again at the point Q, where this perpendicular meets the focal radius, draw CQ perpendicular to it, then C will be the centre of curvature, and CP the radius of curvature.

Another useful construction is founded on the principle that *if a circle intersect a conic, its chords of intersection will make equal angles with the axis.* For, the rectangles under the segments of the chords are equal (Euc. III. 35), and therefore, the parallel diameters of the conic are equal (Art. 147), and, therefore, make equal angles with the axis (Art. 156).

Now in the case of the circle of curvature, the tangent at T (see figure, p. 205) is one chord of intersection, and the line TL the other; we have, therefore, only to draw TL, making the same angle with the axis as the tangent, and we have the point L; then the circle described through the points T, L, and touching the conic at T, is the circle of curvature.

This construction shows that the osculating circle at either vertex has a contact of the third degree.

248. The radius of curvature of the parabola is found by the same method.

The equation, referred to any diameter and tangent ($y^2 = p'x$), is transferred to the tangent and normal by the same substitution as in Art. 246, and we find

$$R = \frac{p'}{2\sin\theta},$$

or since (Arts 214, 215)

$$N = \frac{p'}{2}\sin\theta, \qquad R = \frac{N}{\sin^2\theta} = \frac{N}{\cos^2\psi}.$$

The construction, therefore, used in the last Article, applies also to the case of the parabola.

Ex. 1. *In all the conic sections the radius of curvature is equal to the cube of the normal divided by the square of the semi-parameter.*

Ex. 2. *Express the radius of curvature of an ellipse in terms of the angle which the normal makes with the axis.*

Ex. 3. *Find the lengths of the chords of the circle of curvature which pass through the centre or the focus of a central conic section.*

They will be found $= \frac{2b'^2}{a'}$, and $\frac{2b'^2}{a}$.

Ex. 4. *The focal chord of curvature of any conic is equal to a focal chord of the conic drawn parallel to the tangent at the point.*

Ex. 5. *In the parabola the focal chord of curvature is equal to the parameter of the diameter passing through the point.*

CHAPTER XIII.

METHODS OF ABRIDGED NOTATION.

249. If $S = 0$ and $S' = 0$ be the equations of two conic sections, then (page 48) $S - kS' = 0$ (where k is any constant) is the equation of another conic passing through the real or imaginary points of intersection of S and S'. These points are in number *four*, for we proved (Art. 18) that we obtain an equation of the mn^{th} degree to determine the co-ordinates of the points of intersection of two curves of the m^{th} and n^{th} degrees; but since an equation of the mn^{th} degree has always mn roots, real or imaginary, we infer (as in Art. 73) that a curve of the m^{th} degree will *always* intersect a curve of the n^{th} degree in mn points,* and in the particular case where $m = n = 2$, that two *conic sections always* intersect each other in four points, real or imaginary.

250. Before we proceed to the investigation of the properties of curves represented by the equation $S - kS' = 0$, we wish to call the reader's attention to some of the particular equations included under this general form.

* It follows hence, conversely, that if we could tell the number of real and imaginary points in which any right line is cut by a given curve, we should know the degree of the curve. We shall apply this principle, in Part III., to the investigation of loci.

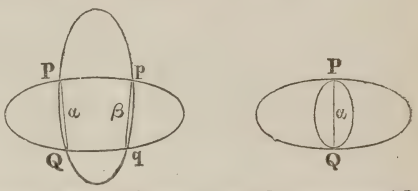
I. If the equation S′ be resolvable into two factors, and represent the two right lines $\alpha = 0$, $\beta = 0$, our equation will take the form

$$S - k\alpha\beta = 0.$$

This equation, which must evidently be satisfied by the co-ordinates of the points where either α or β meets S, is, therefore, the equation of *a conic having the lines* α *and* β *for its chords of intersection with* S. If either α or β do not meet S in real points, it must still be considered as *a chord of imaginary intersection*, and will preserve many important properties in relation to the two curves, as we have already seen in the case of the circle (Art. 108). We have already met with a particular case of the equation $S - k\alpha\beta = 0$, when the axes ($x = 0$, $y = 0$) are the two chords of intersection (see Ex. 3, p. 200, where we proved that the equations of two conic sections will only differ in the co-efficient of xy, if their chords of intersection be the axes of co-ordinates).

II. If the lines α, β coincide, that is, if S′ be a perfect square, the equation becomes

$$S - k\alpha^2 = 0.$$



The points P, p; Q, q, coincide, and the equation represents *a conic having double contact with* S, and whose chord of contact is α.

If, however, the line α *be a tangent to* S, the two points P and Q will also coincide, and *the conics*, having four consecutive points in common, *will have with each other a contact of the third degree* (see Art. 244, where this was proved, in the particular case where α is the axis of x, for we there showed that the equations of two such conics will only differ in the co-efficient of y^2).

III. The forms just given receive important modifications, if either of the lines which they contain be altogether at an infinite distance. We proved (Art. 64) that when a line is removed to an infinite distance, its equation is reduced to the constant term. Hence, if the line β be altogether at an infinite distance, the equation $S - k\alpha\beta$ becomes

$$S - k\alpha\beta' = 0$$

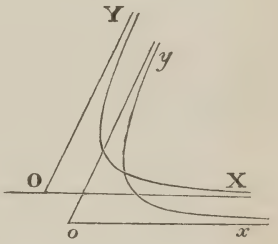
(where β' is a constant which, although we might combine with k into a single constant, yet we prefer writing in such a form as

will make the analogy with I. more apparent). This is, therefore, the equation of a conic intersecting S in the two points where α meets it, and also in the two real, imaginary, or coincident points, in which a line at infinity (β') meets the curve.

It is evident that the curves S and $S - k\alpha\beta'$ must have the quadratic terms in their equations the same, since $k\alpha\beta'$ cannot contain the terms x^2, xy, or y^2; hence these equations represent *conics similar and similarly placed* (Art. 239). We learn, therefore, that *two conics, similar and similarly placed, can only cut each other in two finite points*, and that this is because *they also cut each other in two real, coincident, or imaginary points at infinity.*

251. It may be satisfactory to the learner to see the last result confirmed by geometrical considerations.

First. If the curves be hyperbolæ. The asymptotes of similar hyperbolæ are parallel (Art. 240), that is, they intersect each other at infinity; but each asymptote intersects its own curve at infinity; hence we infer that similar and similarly placed hyperbolæ intersect each other in the two points at infinity, where each is intersected by its own asymptotes (see the figure, where the two hyperbolæ evidently tend to intersect at the two points at infinity, where OX meets ox, and OY meets oy).



Secondly. If the curves be ellipses. Ellipses only differ from hyperbolæ in having imaginary instead of real asymptotes. The directions of the points at infinity on either of two similar ellipses are determined from the same equation ($Ax^2 + Bxy + Cy^2 = 0$) (Arts. 127 and 239). Now, although the roots of this equation are in both cases imaginary, yet they are in both cases *the same* imaginary roots; we infer, therefore, that two similar ellipses pass through the same two imaginary points at infinity.

Thirdly. If the curves be parabolæ. The equation which determines the direction of the point at infinity on a parabola is a perfect square (Art. 129); the two points at infinity, therefore, coincide; and from our definition of a tangent (Art. 83) we learn, that *every parabola has one tangent altogether at an infinite distance*,

an important theorem, of which we shall make frequent use hereafter, and which we shall otherwise arrive at (Art. 255, IX.) The direction of the point at infinity is the same as that of the diameters of the parabola (Art. 136), and is, therefore, the same for two similarly placed parabolæ (Art. 241). Hence, *two similarly placed parabolæ* intersect in two coincident points at infinity, or, in other words, they *touch each other at infinity.*

252. IV. We return to examine the form assumed by the equation $S - ka^2 = 0$, when the line a is at an infinite distance. It then becomes $$S - ka'^2 = 0$$ (where k and a are both constant, which, as in Art. 250, we write separately for the sake of analogy). This equation, therefore, being a particular case of II., represents two conics having double contact with each other, both points of contact being on the line a', that is to say, at an infinite distance. If S be a parabola, then, by the last Article, the line (a') at infinity touches it, and, therefore, the curve $S - ka'^2$ will have with it a contact of the third order at infinity.

Now, if the equation of two conics only differ in the constant terms, since the co-ordinates of the centre (Art. 134) do not contain F, they must have the same centre; and since the first three terms are the same, they must be similar (Art. 239): hence the equation $(S - ka'^2 = 0)$ represents a conic *similar and concentric* with S. We learn, therefore, that *similar and concentric conics may be considered as touching each other in two points at an infinite distance.* This is otherwise evident if the curves be *hyperbolæ*, for, as we proved before, they pass through the same points at infinity, and, since they have the same asymptotes, they have also the same tangents at those points. In like manner, similar and concentric ellipses pass through the same imaginary points at infinity; and since they have the same imaginary asymptotes, they have the same imaginary tangents at those points.

If the curve S be a parabola, then $S - ka'^2$ will be an *equal* parabola, having its axis parallel to that of S, for evidently the equation $y^2 = p\left(x - \frac{ka'^2}{p}\right)$ represents a parabola equal to $y^2 = px$ and similarly placed; and if the origin be transferred to any other

point, the equations will still continue to differ only in the constant term. We learn, therefore, that *two similarly placed and equal parabolæ may be considered as having with each other a contact of the third order at an infinitely distant point.*

253. Since all circles are similar curves, it follows, as a particular case of the last Articles, that *all circles pass through the same two imaginary points at infinity*, and that *concentric circles touch each other in two imaginary points at infinity.* Thus we see the reason why two circles cannot cut each other in more than two finite points, and why two concentric circles do not meet in any finite point, although two curves of the second degree in general intersect in four points. We shall also show that the theorems established (p. 106, &c.), concerning circles which pass through the same two points, are only particular cases of more general theorems concerning conic sections which pass through the same four points.

254. V. If both S and S′ break up into factors, the equation will then be of the form

$$\alpha\gamma - k\beta\delta = 0.$$

This is the equation of *a conic circumscribing the quadrilateral* $(\alpha\beta\gamma\delta)$, for it is evidently satisfied by any of the four suppositions

$$(\alpha = 0,\ \beta = 0),\quad (\beta = 0,\ \gamma = 0),\quad (\gamma = 0,\ \delta = 0),\quad (\delta = 0,\ \alpha = 0).$$

VI. If any of these four lines be at an infinite distance, the equation will take the form

$$\alpha\gamma - k\beta\delta' = 0;$$

α, γ, will then be two lines which meet the curve at infinity, and β will be the line joining the finite points where α, γ, meet the curve. Thus, for example, in the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

if we denote by α, γ, the two lines signified by $Ax^2 + Bxy + Cy^2 = 0$, the equation will be in the form we are now considering. We see, therefore, that α, γ, will meet the curve at infinity, and that $Dx + Ey + F = 0$ is the line joining the finite points where they meet the curve.

255. VII. If S break up into factors while S′ is a perfect square, the equation will be of the form

$$a\gamma - k\beta^2 = 0.$$

This represents *a conic to which* a *and* γ *are tangents, while* β *is the chord joining their points of contact.* For we saw that a meets the curve $(a\gamma - k\beta\delta)$ in the two points $(a\beta)$, $(a\delta)$; now let β and δ coincide, and a meets the curve in two coincident points at $(a\beta)$; that is to say, a is a tangent, and $(a\beta)$ its point of contact. In like manner, γ is a tangent, and $(\beta\gamma)$ its point of contact.

VIII. If the line β be at an infinite distance, the equation becomes

$$a\gamma - k\beta'^2 = 0;$$

a, γ, are therefore tangents, whose point of contact is infinitely distant, that is to say, they are asymptotes to the curve. We met with a particular case of this (Art. 161) in the equation

$$xy = \mathrm{F},$$

where we proved that the lines $x = 0$, $y = 0$, were asymptotes to the curve.

IX. If either of the lines a, γ, be at an infinite distance, the equation becomes

$$a\gamma' - k\beta^2 = 0;$$

then, since the highest terms form the perfect square β^2, the curve will be a parabola; the curve will have one tangent γ' at an infinite distance; a will be another tangent, and β will be the line joining the points of contact of these tangents. Now, since it passes through the point of contact of the tangent at infinity, it must be a diameter of the curve (Art. 136); β is, therefore, the diameter drawn through the point of contact of the tangent a.

We had a particular example of this in the equation

$$y^2 = px.$$

The line p at infinity is one tangent, the line x another, and the line y is a diameter through the point of contact of x.

More generally,

$$(ax + by)^2 + \mathrm{D}x + \mathrm{E}y + \mathrm{F} = 0,$$

is the equation of a parabola to which $\mathrm{D}x + \mathrm{E}y + \mathrm{F}$ is a tangent, and $ax + by$ the diameter through the point of contact.

X. We may notice here one or two forms which are easily reducible to the form $a\gamma = k\beta^2$. For example,

$$a^2 - pa\beta + q\beta^2 = k\gamma^2$$

may be put into the form

$$(a - \mu\beta)\ (a - \mu'\beta) = k\gamma^2$$

(μ and μ' being the roots of a quadratic). Therefore, $a - \mu\beta$ and $a - \mu'\beta$ are tangents at the extremities of the chord γ, and since these tangents evidently pass through the point $(a\beta)$, this point is the pole of the line γ, with regard to the conic. If the roots μ and μ' be imaginary, $(a\beta)$ is still the pole of γ, but it lies inside the conic, and the tangents through it are imaginary.

XI. Again, the equation

$$l^2a^2 + m^2\beta^2 = n^2\gamma^2,$$

is that of a conic such that any of the lines a, β, γ, *is the polar of the point of intersection of the other two,* for the equation may be written in any of the forms

$$(n\gamma - la)\ (n\gamma + la) = m^2\beta^2,$$
$$(n\gamma - m\beta)\ (n\gamma + m\beta) = l^2a^2,$$
$$(la - m\beta\sqrt{-1})\ (la + m\beta\sqrt{-1}) = n^2\gamma^2;$$

therefore, tangents at the extremities of β pass through $(a\gamma)$; at the extremities of a pass through $(\beta\gamma)$; while the tangents at the imaginary points where γ meets the conic pass through $(a\beta)$. We had a particular case of this form in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the lines $a\beta$ being here the axes, and the line γ at infinity, whence it may be seen, that the point (xy) is the polar of the line at infinity, and that the pole of each axis is a point at infinity on the other.

256. XII. If both S and S′ be perfect squares, the equation is of the form

$$a^2 - k\gamma^2 = 0,$$

and represents the two right lines $a \pm \gamma\sqrt{k} = 0$. These lines will evidently be imaginary if k be negative, and parallel if γ be at an infinite distance.

XIII. In general, if we wished to find the condition that the

equation $S - kS' = 0$ should represent two right lines, we may write the equation at full length,

$$(A - kA')x^2 + (B - kB')xy + (C - kC')y^2 + (D - kD')x + (E - kE')y + F - kF' = 0,$$

and then applying the condition in Art. 150, we shall obtain a cubic to determine k. This is otherwise evident, since through the four points of intersection of S and S′ *three* distinct pairs of right lines can be drawn, viz. (see figure, p. 210) the three pairs of lines Pp, Qq; PQ, pq; Pq, pQ.

257. *To find the condition that two conics should touch, whose equations are given.*

We have only to arrange the equation just found in powers of k, and to form the condition that this cubic should have two equal roots, and we shall have an equation which must be satisfied when the two conics touch. For if the points P, p coincide, the last two pairs of lines become identical, and consequently the cubic which determines k must in this case have two equal roots. Now the condition that the cubic

$$Lk^3 + Mk^2 + Nk + P = 0$$

should have two equal roots is readily found to be

$$(MN - 9LP)^2 = 4(M^2 - 3LN)(N^2 - 3MP).$$

Substituting for L, M, N, P, their values in terms of the coefficients of the given equations, it will be seen that the required condition is of the sixth degree in terms of the coefficients of each equation.

It follows hence that the problem "to describe a conic through four given points to touch a given conic," admits in general of six solutions.

258. Having now enumerated the principal forms of equations which we shall have occasion to use, we proceed to give some theorems immediately suggested by the geometrical interpretation of these equations.

If $S = 0$ be the equation to a circle, then (Art. 93) S is the square of the tangent from any point xy to the circle: hence, by Art. 31, $S - k\alpha\beta = 0$ (the equation of a conic whose chords of

intersection with the circle are α and β) expresses that *the locus of a point, such that the square of the tangent from it to a fixed circle is in a constant ratio to the product of its distances from two fixed lines, is a conic passing through the four points in which the fixed lines intersect the circle.*

This theorem is equally true whatever be the magnitude of the circle, and whether the right lines meet the circle in real or imaginary points; thus, for example, if the circle be infinitely small, *the locus of a point, the square of whose distance from a fixed point is in a constant ratio to the product of its distances from two fixed lines, is a conic section;* and the fixed lines may be considered as chords of imaginary intersection of the conic with an infinitely small circle whose centre is the fixed point.

259. Similar inferences can be drawn from the equation $S - ku^2 = 0$, where S is a circle. We learn that *the locus of a point, such that the tangent from it to a fixed circle is in a constant ratio to its distance from a fixed line, is a conic touching the circle at the two points where the fixed line meets it;* or, conversely, that *if a circle have double contact with a conic, the tangent drawn to the circle from any point on the conic is in a constant ratio to the perpendicular from the point on the chord of contact.*

In the particular case where the circle is infinitely small, we obtain the fundamental property of the focus and directrix, and we infer that *the focus of any conic may be considered as an infinitely small circle, touching the conic in two imaginary points situated on the directrix.*

260. In general, *if in the equation of any conic the co-ordinates of any point be substituted, the result will be proportional to the rectangle under the segments of a chord drawn through the point parallel to a given line.**

For (Art. 146) this rectangle

$$= \frac{F'}{A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta},$$

where, by Art. 126, F′ is the result of substituting in the equation the co-ordinates of the point; if, therefore, the angle θ be con-

* This is equally true for curves of any degree.

stant, this rectangle will be proportional to F'. Hence we may extend the last proved theorems to the case where S is any conic; for example: "If two conics have double contact, the square of the perpendicular from any point of one upon the chord of contact, is in a constant ratio to the rectangle under the segments of that perpendicular made by the other;" or, in general, "if a line parallel to a given one meet two conics in the points P, Q. p, q, and we take on it a point O, such that the rectangle OP.OQ may be to $Op.Oq$ in a constant ratio, the locus of O is a conic through the points of intersection of the given conics."

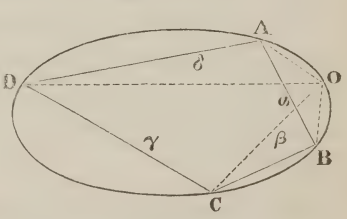
261. The equation (V.) $\alpha\gamma = k\beta\delta$, similarly interpreted, leads to the important theorem: *The product of the perpendiculars let fall from any point of a conic on two opposite sides of an inscribed quadrilateral is in a constant ratio to the product of the perpendiculars let fall on the other two sides.*

From this property we at once infer, that *the anharmonic ratio of a pencil, whose sides pass through four fixed points of a conic, and whose vertex is any variable point of it, is constant.*

For the perpendicular

$$\alpha = \frac{OA \,.\, OB \,.\, \sin AOB}{AB}, \qquad \gamma = \frac{OC \,.\, OD \,.\, \sin COD}{CD}, \text{ \&c.}$$

Now if we substitute these values in the equation $\alpha\gamma = k\beta\delta$, the continued product OA. OB. OC. OD will appear on both sides of the equation, and may therefore be suppressed, and there will remain



$$\frac{\sin AOB \,.\, \sin COD}{\sin BOC \,.\, \sin AOD} = k \,.\, \frac{AB \,.\, CD}{BC \,.\, AD};$$

but the right hand member of this equation is constant, while the left hand member is the anharmonic ratio of the pencil OA, OB, OC, OD.

The consequences of this theorem are so numerous and important, that we shall devote a section of the next chapter to develope them more fully.

262. The equation (VII.) $\alpha\gamma - k\beta^2 = 0$, similarly interpreted,

proves that *the product of the perpendiculars from any point of a conic upon two fixed tangents is in a constant ratio to the square of the perpendicular on the chord of contact.*

From the equation (VI.) $a\gamma - k\beta\delta = 0$ we learn that "the product of the perpendiculars from any point of a hyperbola, on two fixed parallels to the asymptotes, is proportional to the perpendicular on the line joining the points where they meet the curve."

And from the equation (VIII.) $a\gamma = k\beta^2$, *that the product of the perpendiculars from any point of the curve on the asymptotes is constant.*

263. *If two conics have each double contact with a third, their chords of contact with the third conic, and a pair of their chords of intersection with each other, will all pass through the same point, and will form an harmonic pencil.*

Let the equation of the third conic be $S = 0$, and those of the other two conics,

$$S - L^2 = 0, \qquad S - M^2 = 0.$$

(The reader will perceive the reason why we use the Roman letters, on comparing this form with that given in Art. 250, II., and on referring to Art. 50).

Now, on subtracting these equations, we find for the equation of the chords of intersection,

$$L^2 - M^2 = 0.$$

The chords of intersection, therefore ($L - M = 0$, $L + M = 0$), pass through the intersection of the chords of contact (L and M), and form an harmonic pencil with them (Art. 54).

It is important that the student should acquire the habit of taking notice of the number of particular theorems often included under one general enunciation; thus, for example, the present theorem holds good, and is proved, in like manner, if the conic S reduce to two right lines; hence, *the chords of contact of two conics with their common tangents pass through the intersection of their common chords.*

Again, if S be any conic, while $S - L^2$ and $S - M^2$ both reduce to pairs of right lines, these right lines will then form a circumscribing quadrilateral, and the chords of intersection ($L^2 - M^2$)

will be the diagonals of that quadrilateral, while the chords of contact (L and M) obviously are the diagonals of the inscribed quadrilateral formed by joining the points of contact. Hence, *the diagonals of any inscribed, and of the corresponding circumscribed quadrilateral, pass through the same point, and form an harmonic pencil.*

The theorem of this Article may also be stated thus: *If a conic section pass through two given points, and have double contact with a given conic, the chord of contact passes through a fixed point.* For, suppose any conic ($S - L^2 = 0$) through the two given points to be fixed, then the intersection of its chord of contact (L) with the line joining the given points, determines a point through which, by the present Article, any other chord of contact must pass.

In like manner: *Given two tangents and two points on a conic section, the chord of contact will pass through a fixed point on the line joining the two given points.*

264. *If three conics have each double contact with a fourth, their six chords of intersection will pass three by three through the same points,* thus forming the sides and diagonals of a quadrilateral.

Let the conics be

$$S - L^2 = 0, \qquad S - M^2 = 0, \qquad S - N^2 = 0.$$

By the last Article the chords will be

$$\begin{aligned} &L - M = 0, \quad M - N = 0, \quad N - L = 0; \\ &L + M = 0, \quad M + N = 0, \quad N - L = 0; \\ &L + M = 0, \quad M - N = 0, \quad N + L = 0; \\ &L - M = 0, \quad M + N = 0, \quad N + L = 0. \end{aligned}$$

As in the last Article, we might deduce hence many particular theorems, by supposing one or more of the conics to break up into right lines.

Thus, for example, if both S and $S - L^2$ break up into right lines (which will happen when L denotes a right line passing through the intersection of the two right lines denoted by S (see Art. 130)), S represents two common tangents to $S - M^2$, $S - N^2$, while $S - L^2$ represents any two right lines passing through the intersection of those common tangents. Hence, *if through the in-*

tersection of the common tangents of two conics we draw any pair of right lines, the chords of each conic joining the extremities of those lines will meet on one of the common chords of the conics. This is the extension of Art. 117. Or, again, *tangents at the extremities of either of these right lines will meet on one of the common chords.*

Again, if $S - L^2$, $S - M^2$, $S - N^2$, all break up into pairs of right lines, they will form a hexagon circumscribing S, the chords of intersection will be diagonals of that hexagon, and the proposition of this Article becomes Brianchon's theorem: "*The three opposite diagonals of every hexagon circumscribing a conic intersect in a point.*"

By the *opposite* diagonals we mean (if the sides of the hexagon be numbered 1, 2, 3, 4, 5, 6) the lines joining (1, 2) to (4, 5), (2, 3) to (5, 6), and (3, 4) to (6, 1); and by changing the order in which we take the sides, we may consider the same lines as forming a number (sixty) of different hexagons, for each of which the present theorem is true.

By supposing two sides of the hexagon to be indefinitely near, we obtain from this theorem a very simple construction,—"Given five tangents, to find the point of contact of any of them,"—since any tangent is intersected by a consecutive tangent at its point of contact (Art. 130).

265. *If three conic sections have one chord common to all, their three other common chords will pass through the same point.*

Let the equation of one be $S = 0$, and of the common chord $L = 0$, then the equations of the other two are of the form

$$S - LM = 0, \qquad S - LN = 0,$$

which must have, for their intersection with each other,

$$L(M - N) = 0;$$

but $M - N$ is a line passing through the point (MN).

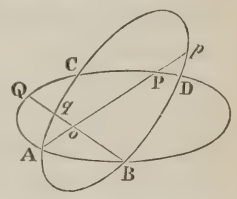
According to the remark in Art. 253, this is only an extension of the theorem (Art. 110), that the radical axes of three circles meet in a point. For three circles have one chord (the line at infinity) common to all, and the radical axes are their other common chords.

The first theorem of the last Article may be considered as a

still further extension of the same theorem, and three conics which have each double contact with a fourth may be considered as having four radical centres, through each of which pass three of their common chords.

The theorem of this Article may, as in Art. 110, be otherwise enunciated: *Given four points on a conic section, its chord of intersection with a fixed conic passing through two of these points will pass through a fixed point.*

A number of particular inferences may also be drawn from the theorem of the present Article, by supposing one or more of the conics to break up into two right lines. Thus, for example, if one of the conics break up into the pair of lines OA, OB, we obtain the theorem: "If through one of the points of intersection of two conics we draw any line meeting the conics in the points P, p, and through any other point of intersection B a line meeting the conics in the points Q, q, then the lines PQ, pq, will meet on CD, the other chord of intersection." Next let the points A, B, coincide, then the two conics will touch at A, and we learn that "if two right lines, drawn through the point of contact of two conics, meet the curves in points P, p, Q, q, then the chords PQ, pq, will meet on the chord of intersection of the conics."



This is a particular case of a theorem given in the last Article, since one intersection of common tangents to two conics which touch, reduces to the point of contact (Art. 119).

This theorem is equally true, if the conics have contact with each other of the second or third order (see Art. 244, where we showed that the common chord passes through the point of contact in the first case, and coincides with the tangent in the second).

266. The equation of a conic circumscribing a given quadrilateral,

$$a\gamma = k\beta\delta,$$

enables us to write down the equation of a conic passing through five points.

For, being given the co-ordinates of four of them, we can, by Art. 33, form the equations α, β, γ, δ, and then, if we substitute

in this equation the co-ordinates of the fifth point, we obtain $k = \frac{a'\gamma'}{\beta'\delta'}$, where $a'\beta'\gamma'\delta'$ are the results of this substitution in $a\beta\gamma\delta$.

This equation also furnishes us with a proof of "Pascal's theorem," that *the three intersections of the opposite sides of any hexagon inscribed in a conic section are in one right line.*

Let the vertices be *abcdef*, and let *ab* denote the equation of the line joining the points *a*, *b*, then since the conic circumscribes the quadrilateral *abcd*, its equation must be capable of being put into the form

$$ab.cd - bc.ad = 0.$$

But since it also circumscribes the quadrilateral *defa*, the same equation must be capable of being expressed in the form

$$de.fa - ef.ad = 0.$$

From the identity of these expressions we have

$$ab.cd - de.fa = (bc - ef)\,ad.$$

Hence we learn that the left hand side of this equation (which from its form represents a figure circumscribing the quadrilateral formed by the lines *ab*, *cd*, *de*, *af*) is resolvable into two factors, which must therefore represent the diagonals of that quadrilateral. But *ad* is evidently the diagonal which joins the vertices *a* and *d*, therefore $bc - ef$ must be the other, and must join the points (ab, cd), (de, af); and since from its form it denotes a line through the point (bc, ef) it follows that these three points are in one right line.

We shall in the next chapter give another demonstration of this important theorem, which, as we shall prove in the next Part, may also be derived as a simple consequence of a general property of lines of the third degree.

By supposing two vertices of the hexagon to be indefinitely near, we may, "given five points on a conic, draw a tangent at any of these points."

267. We may, as in the case of Brianchon's theorem, obtain a number of different theorems concerning the same six points, according to the different orders in which we take them. Thus since the conic circumscribes the quadrilateral *bcef*, its equation can be expressed in the form

$$be.cf - bc.ef = 0.$$

Now, from identifying this with the first form given in the last Article, we have

$$ab \,.\, cd - be \,.\, cf = (ad - ef)\, bc;$$

whence, as before, we learn that the three points (ab, cf), (cd, be), (ad, ef) lie in one right line, viz., $ad - ef = 0$.

In like manner, from identifying the second and third forms of the equation of the conic, we learn that the three points (de, cf), (fa, be), (ad, bc) lie in one right line, viz., $bc - ad = 0$. But the three right lines

$$bc - ef = 0, \quad ef - ad = 0, \quad ad - bc = 0,$$

meet in a point (Art. 51). Hence we have Steiner's theorem, that "the three Pascal's lines meet in a point which are obtained by taking the vertices in the orders respectively, *abcdef*, *adcfeb*, *afcbed*." For some further developments on this subject we refer the reader to the note at the end of Chapter XIV.

TRILINEAR CO-ORDINATES.

268. The few examples we have given are sufficient to convince the learner of the necessity of attending to the interpretation of the equations used in analytic geometry, and of the importance of taking notice of the theorems often spontaneously presented to us by the form of our equations. The beginner may, however, find it difficult to apply these methods, when the object is not to find what theorems can be deduced from a given equation, but to discover what equations will demonstrate a given theorem. We proceed, therefore, to show that in this latter case these methods will furnish solutions, although not so rapid as in the former case, yet more concise than those obtained by ordinary analytic geometry.

We proved (Art. 62) that, being given the equations of three lines (α, β, γ), there is no line whose equation may not be put into the form

$$A\alpha + B\beta + C\gamma = 0,$$

(A, B, and C, being constants).

We can show, in like manner, that there is no conic section whose equation may not be written in the form

$$A\alpha^2 + B\alpha\beta + C\beta^2 + D\alpha\gamma + E\beta\gamma + F\gamma^2 = 0,$$

for this equation is of the second degree, and since it contains the same number of independent constants (five) as the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

it is equally capable of representing any particular conic.

In like manner, we can show that there is no curve of any degree whose equation may not be expressed as a homogeneous function of the quantities α, β, γ.

For it can be readily proved that the number of terms in the *complete* equation of the n^{th} order, between *two* variables, $\left(= \frac{(n+1)\ (n+2)}{1.2}\ \text{(Art. 78)}\right)$, is the same as the number of terms in the *homogeneous* equation of the n^{th} order between *three* variables.

The methods used (Arts. 59, 102, &c.) may be considered, at pleasure, either as abbreviated forms of equations expressed in *Cartesian* (or x and y) co-ordinates, or else as an independent system of *trilinear* co-ordinates, in which the position of any point is expressed by its distances from three fixed right lines, α, β, γ. The advantage of trilinear co-ordinates is, that whereas, in Cartesian co-ordinates, the utmost simplification we can introduce is by choosing *two* of the most remarkable lines in the figure for our axes of co-ordinates, we can, in trilinear co-ordinates, obtain still more simple expressions, by choosing *three* of the most remarkable lines for the lines of reference, α, β, γ. We shall show hereafter that this advantage is perfectly analogous to that gained by the use of *the method of projections*, and that, except in particular cases, both methods prove too feeble when applied to curves of high dimensions, whose equations remain too complicated even after this simplification.

269. Before applying this method to examples, we wish to show that Cartesian co-ordinates are only a particular case of trilinear. There appears, at first sight, to be an essential difference between them, since trilinear equations are always homogeneous, while we are accustomed to speak of Cartesian equations as containing an absolute term, terms of the first degree, terms of the second degree, &c. A little reflection, however, will show that this difference is only apparent, and that Cartesian equations must

be equally homogeneous in reality, though not in form. The equation $x = 3$, for example, must mean that the line x is equal to three feet or three inches, or, in short, to three times some linear unit; the equation $xy = 9$ must mean that the rectangle xy is equal to nine *square* feet or *square* inches, or *to the square* of some linear unit; and, similarly, in the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

if the co-efficients A, B, C, be numbers, the co-efficients D and E must be multiples of some linear unit, and the co-efficient F a multiple of the square of that unit. If we wish to have our equation homogeneous in form as well as in reality, we may denote our linear unit by k, and write the equation in the form

$$Ax^2 + Bxy + Cy^2 + Dxk + Eyk + Fk^2 = 0. \qquad (1)$$

Now, if we compare this equation with the equation

$$A\alpha^2 + B\alpha\beta + C\beta^2 + D\alpha\gamma + E\beta\gamma + F\gamma^2 = 0; \qquad (2)$$

and remember (Art. 64) that when a line is altogether at an infinite distance its equation takes the form $k = 0$, we learn that *equations in Cartesian co-ordinates are only the particular form assumed by trilinear equations when two of the lines of reference are what are called the co-ordinate axes, while the third is at an infinite distance.*

270. We shall find it of the greatest advantage to keep constantly in view the analogy which subsists between these two forms of equations. If, for instance, we make $\gamma = 0$ in equation (2), the result $A\alpha^2 + B\alpha\beta + C\beta^2$ is plainly the equation of the lines joining the point $(\alpha\beta)$ to the points where γ cuts the curve. So, in like manner, if we make $k = 0$ in equation (1), the result $(Ax^2 + Bxy + Cy^2 = 0)$ must be the equation of the line joining the origin (xy) to the points where the line at infinity (k) cuts the curve (Art. 127).

In like manner, the equation of any curve may be written

$$u_n + u_{n-1}k + u_{n-2}k^2 + u_{n-3}k^3 +, \text{ \&c.}$$

(where we use the abbreviations u_n, u_{n-1}, &c. to denote terms of the n^{th}, $n - 1^{st}$, &c., degrees).

Now, if we seek the points where the line at infinity meets the curve, we have only to make $k = 0$, and we obtain the equa-

tion $u_n = 0$; hence we infer that the directions of the points at infinity on any curve are found by putting the highest terms of the equation = 0.

Again, we saw (Art. 129), that, if in the equation of the second degree A = 0, the axis of x will meet the curve in one infinitely distant point. The same thing appears, by making $y = 0$ in the equation, which will then reduce to

$$Dxk + Fk^2 = 0.$$

Now this is not to be considered as a simple equation, but as the product of the two equations $k = 0$ and $Dx + Fk = 0$. The axis, therefore, meets the curve, not only in the finite point where it meet the line $(Dx + F)$, but also in the point at infinity where it meets the line k.

In like manner, if both A and D = 0, the points where the axis meets the curve are given by the equation $Fk^2 = 0$; hence, the axis meets the curve in two coincident points at infinity, and is, therefore, an asymptote.

This is the most simple explanation of the difficulty noticed in Art. 127. The same explanation may be given in other cases in which an equation appears to lose dimensions.

271. We shall commence our examples of the use of trilinear co-ordinates with the equation of a conic section, referred to two tangents and their chord of contact,

$$LM = R^2$$

(Arts. 50 and 255), and shall first show how the equation of any line connected with the conic can be expressed in terms of L, M, R.

We can express the position of any point on the curve by a *single* variable (Art. 175); for if

$$\mu L = R$$

be the equation of the line joining any point on the curve to (LR), then, substituting in the equation of the curve, we get

$$M = \mu R \text{ and } \mu^2 L = M$$

for the equations of the lines joining this point to (MR) and (LM): any two of these three equations, therefore, will determine a point on the curve. We shall call this point the point μ.

We can form, by Art. 68, the equation of the line joining two points on the curve μ and μ', and we get

$$\mu\mu'L - (\mu + \mu')R + M = 0,$$

an equation evidently satisfied by either of the suppositions

$$(\mu L = R,\ \mu R = M),\ \text{or}\ (\mu' L = R,\ \mu' R = M).$$

If μ and μ' coincide, we find *the equation of the tangent*, viz.,

$$\mu^2 L - 2\mu R + M = 0.$$

Hence, conversely, if the equation of a right line $(\mu^2 L - 2\mu R + M = 0)$ contain an indeterminate quantity μ *in the second degree*, the right line will always touch a conic section $(LM = R^2)$ (Art. 69).

272. *Given four points of a conic, the anharmonic ratio of the pencil joining them to any fifth point is constant.*

Let the points be (LR), (MR), μ', μ'', then the four lines joining them to any fifth point μ, are

$$\mu L - R = 0, \qquad M - \mu R = 0,$$
$$\mu'(\mu L - R) + (M - \mu R) = 0, \qquad \mu''(\mu L - R) + (M - \mu R) = 0,$$

and their anharmonic ratio is (Art. 54) $= \frac{\mu'}{\mu''}$, and is, therefore, independent of the position of the point μ.

We shall in future, for brevity, use the expression, "the anharmonic ratio of four points of a conic," when we mean the anharmonic ratio of a pencil joining those points to any fifth point on the curve. We can express the anharmonic ratio of the four points μ', μ'', μ''', μ''''; for since LR is a point on the conic, it is that of the lines joining those points to the point LR, whose equations are

$$\mu' L - R = 0,\ \ \mu'' L - R = 0,\ \ \mu''' L - R = 0,\ \ \mu'''' L - R = 0,$$

and the anharmonic ratio is (Art. 55)

$$\frac{(\mu' - \mu'')(\mu''' - \mu'''')}{(\mu' - \mu''')(\mu'' - \mu'''')},$$

a form easily remembered from its analogy to the anharmonic ratio of four points in a right line.

Four fixed tangents cut any fifth in points whose anharmonic ratio is constant.

Let the fixed tangents be those at the points μ', μ'', μ''', μ'''';

and the variable tangent that at the point μ, then the anharmonic ratio in question is the same as that of the pencil joining the four points of intersection to the point LM. Now if we eliminate R from the equations of any two tangents,

$$\mu^2 L - 2\mu R + M = 0,$$
$$\mu'^2 L - 2\mu' R + M = 0,$$

we obtain $$\mu\mu' L - M = 0,$$

the equation of the line joining LM to the intersection of these two tangents. The anharmonic ratio in question is therefore that of the four lines,

$$\mu\mu' L - M = 0,\ \mu\mu'' L - M = 0,\ \mu\mu''' L - M = 0,\ \mu\mu'''' L - M = 0,$$

which by Art. 55 is

$$\frac{(\mu' - \mu'')}{(\mu' - \mu''')}\frac{(\mu''' - \mu'''')}{(\mu'' - \mu'''')},$$

a result independent of μ. Hence too we see that the anharmonic ratio of four tangents is the same as that of their points of contact.

273. Since the equation of the line joining any point to (LM) is $\mu^2 L - M$, we see that the two points $+\mu$ and $-\mu$ lie on a right line passing through LM.

The expression given in the last Article for the anharmonic ratio of four points on a conic, μ', μ'', μ''', μ'''', remains unchanged, if we alter the sign of each of these quantities; hence we derive an important theorem, that *if we draw four lines through any point* LM, *the anharmonic ratio of four of the points* $(\mu', \mu'', \mu''', \mu'''')$ *where these lines meet the conic, is equal to the anharmonic ratio of the other four points* $(-\mu', -\mu'', -\mu''', -\mu'''')$ *where these lines meet the conic.*

The equation in this form enables us easily to investigate properties of two conic sections relating to the point of intersection of their common tangents. For, let L and M be common tangents to two conics, and their equations will be

$$LM - R^2 = 0, \qquad LM - R^2 = 0,$$

A point of one conic is said to *correspond* to a point of the other if the line joining them passes through (LM) the intersection of common tangents. This will be the case if they have the

same μ, since the equation $\mu^2 L - M = 0$ does not involve R or R′. Points are said to correspond inversely if they have the same μ with opposite signs. The chord joining any two points of one conic is said to correspond to the chord joining the corresponding points of the other.

Corresponding lines must meet on one or other of the common chords of the curves (Art. 264).

The chords of intersection of $LM - R^2$ and $LM - R'^2$ are

$$R^2 - R'^2 = 0,$$

but

$$\mu\mu' L - (\mu + \mu') R + M = 0,$$
$$\mu\mu' L - (\mu + \mu') R' + M = 0,$$

evidently intersect on the common chord $R - R'$. If the lines correspond inversely, they meet on the common chord $R + R'$, as will be seen by changing the signs of μ and μ' in the latter equation.

The anharmonic ratio of four points of one conic is equal to the anharmonic ratio of the four corresponding points of the other.

This useful theorem follows immediately from the expression for the anharmonic ratio of four points given in the last Article, and from the fact that corresponding points have the same μ.

274. *To find the equation of the polar of any point.*

Let the co-ordinates of the point substituted in the equation of either tangent through it give the result

$$\mu^2 L' - 2\mu R' + M' = 0.$$

Now, at the point of contact, $\mu^2 = \frac{M}{L}$, and $\mu = \frac{R}{L}$ (Art. 271).

Therefore, the co-ordinates of the point of contact satisfy the equation

$$ML' - 2RR' + M'L = 0,$$

which is, therefore, that of the polar required.

We may sometimes express a point by the equations

$$aL - R = 0, \qquad bR - M = 0;$$

in this case, by exactly the same method, the equation of the polar is found to be

$$abL - 2aR + M = 0.$$

275. It is evident that if we were given any relation between the μ's of two points, we could find the envelope of the chord

joining them, or the locus of the intersection of their tangents. One or two simple cases of this are worth mentioning. For example, if we were given the *product* of two μ's, $\mu\mu' = a$, then (Art. 272) the intersection of their tangents will lie on the right line $a\mathrm{L} - \mathrm{M} = 0$; and by substituting a for $\mu\mu'$ in the equation of the chord joining the points, we see that this chord must pass through the fixed point $(a\mathrm{L} + \mathrm{M}, \mathrm{R})$.

In general the chord joining two points,

$$\mu\mu'\mathrm{L} - (\mu + \mu')\,\mathrm{R} + \mathrm{M} = 0,$$

will pass through a fixed point (Art. 71) if

$$a\mu\mu' - b\,(\mu + \mu') + c = 0,$$

where a, b, c, are any constants; that is, if

$$\mu' = \frac{b\mu - c}{a\mu - b}.$$

If the *ratio* of two μ's be given, $\mu' = k\mu$, the equation of the chord becomes

$$k\mu^2\mathrm{L} - (1 + k)\,\mu\mathrm{R} + \mathrm{M} = 0;$$

the chord must, therefore (Art. 271), always touch the conic

$$4k\mathrm{LM} = (1 + k)^2\,\mathrm{R}^2.$$

This property may be expressed in a more symmetrical form, as follows: "The chord joining the joints $\mu \tan\phi$, $\mu \cot\phi$, will always touch the conic $\mathrm{LM} = \dfrac{\mathrm{R}^2}{\sin^2 2\phi}$ at the point μ on that conic." It can be proved, in like manner, that "the locus of the intersection of tangents at the points $\mu \tan\phi$ and $\mu \cot\phi$, will be the conic $\mathrm{LM} = \mathrm{R}^2 \sin^2 2\phi$."

Since the expression for the anharmonic ratio of four points on a conic (Art. 272) remains unaltered, if we multiply each μ either by $\tan\phi$ or by $\cot\phi$, we obtain an important theorem: "*If two conics have double contact, the anharmonic ratio of four of the points in which any four tangents to the one meet the other, is the same as that of the other four points in which the four tangents meet the curve, and also the same as that of the four points of contact.*"*

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* This extension of the theorem in page 229 was communicated to me by Mr. Townsend, who had obtained it geometrically.

Or, again: "If from four points of one of the conics pairs of tangents be drawn to the other, the anharmonic ratio of one set of points of contact is equal to the anharmonic ratio of the other set."

If, in the expression for the anharmonic ratio of four points (Art. 272), we substitute for each μ, $\frac{a + b\mu}{c + d\mu}$ (a, b, c, d, being constants), the anharmonic ratio will remain unaltered. It will be found that this is the most general substitution we can make for μ, which will leave the anharmonic ratio unchanged. The chord joining μ, $\frac{a + b\mu}{c + d\mu}$, will envelope a conic having double contact with the given one. For its equation is

$$\mu(a + b\mu)\,\mathrm{L} - \{(a + b\mu) + \mu(c + d\mu)\}\,\mathrm{R} + (c + d\mu)\,\mathrm{M} = 0,$$

or $$(b\mathrm{L} - d\mathrm{R})\,\mu^2 + (a\mathrm{L} - b\mathrm{R} - c\mathrm{R} + d\mathrm{M})\,\mu + c\mathrm{M} - a\mathrm{R},$$

a line always touching a conic whose equation can be written in the form

$$4(bc - ad)\,(\mathrm{LM} - \mathrm{R}^2) + \{a\mathrm{L} + (b - c)\,\mathrm{R} - d\mathrm{M}\}^2 = 0,$$

and which, therefore, has double contact with the given conic. We may see, from the beginning of this Article, that the touched conic will reduce to a point if $b = -c$.

Hence, "Given three pairs of points on a conic, A, B, C; A′, B′, C′; the envelope of a fourth line DD′, such that the anharmonic ratio of ABCD is equal to that of A′B′C′D′, will be a conic having double contact with the given one."

276. From the preceding Articles we are enabled to apply trilinear co-ordinates to the investigation of any question relating to the *position* of lines (Art. 1). We suppress some formulæ relating to the magnitude of lines and angles, as, where these are concerned, it is in general more advantageous to use ordinary rectangular co-ordinates. In the following examples we have designedly selected questions which seemed least readily to admit of the application of algebraic methods.

Ex. 1. *A triangle is circumscribed to a given conic; two of its vertices move on fixed right lines: to find the locus of the third.*

Let us take for lines of reference the two tangents through the

intersection of the fixed lines, and their chord of contact. Let the equations of the fixed lines be

$$aL - M = 0, \qquad bL - M = 0,$$

while that of the conic is

$$LM - R^2, = 0.$$

Now we proved (Art. 275) that two tangents which meet on $aL - M$ must have the product of their μ's $= a$; hence, if one side of the triangle touch at the point μ, the others will touch at the points $\frac{a}{\mu}$, $\frac{b}{\mu}$, and their equations will be

$$\frac{a^2}{\mu^2}L - 2\frac{a}{\mu}R + M = 0,$$

$$\frac{b^2}{\mu^2}L - 2\frac{b}{\mu}R + M = 0.$$

μ can easily be eliminated from the last two equations, and the locus of the vertex is found to be

$$LM = \frac{4ab}{(a+b)^2}R^2,$$

the equation of a conic having double contact with the given one along the line R.

Ex. 2. *To find the envelope of the base of a triangle, inscribed in a conic, and whose two sides pass through fixed points.*

Take the line joining the fixed points for R, let the equation of the conic be $LM = R^2$, and those of the lines joining the fixed points to LM be

$$aL + M = 0, \qquad bL + M = 0.$$

Now it was proved (Art. 275) that the extremities of any chord passing through $(aL + M, R)$, must have the product of their μ's $= a$.

Hence, if the vertex be μ, the base angles must be $\frac{a}{\mu}$ and $\frac{b}{\mu}$, and the equation of the base must be

$$abL - (a+b)\mu R + \mu^2 M = 0.$$

The base must, therefore (Art. 271), always touch the conic

$$LM = \frac{(a+b)^2}{4ab}R^2,$$

a conic having double contact with the given one along the line joining the given points.

Ex. 3. *To inscribe in a conic section a triangle whose sides pass through three given points.*

Two of the points being assumed, as in the last example, we saw that the equation of the base must be

$$ab\mathrm{L} - (a+b)\,\mu\mathrm{R} + \mu^2\mathrm{M} = 0.$$

Now if this line pass through the point $c\mathrm{L} - \mathrm{R} = 0$, $d\mathrm{R} - \mathrm{M} = 0$, we must have

$$ab - (a+b)\mu c + \mu^2 cd = 0,$$

an equation sufficient to determine μ.

Now at the point μ we have $\mu\mathrm{L} = \mathrm{R}$, $\mu^2\mathrm{L} = \mathrm{M}$; hence the coordinates of this point must satisfy the equation

$$ab\mathrm{L} - (a+b)\,c\mathrm{R} + cd\mathrm{M} = 0.$$

The question, therefore, admits of two solutions, for either of the points in which this line meets the curve may be taken for the vertex of the required triangle.

The solution here given, although algebraically complete, has the disadvantage of not pointing out how to construct geometrically the line whose equation has just been given; it will be a useful exercise, however, on the preceding formulæ, if the student verify by this method the following construction, which we shall prove otherwise in the next chapter: "Form the triangle whose sides are the polars of the three given points, join each point to the opposite vertex of this triangle, and the line joining the points in which two of these lines meet the opposite sides of the polar triangle will be the required line."

The three given points are

$$(a\mathrm{L} + \mathrm{M},\ \mathrm{R}),\quad (b\mathrm{L} + \mathrm{M},\ \mathrm{R}),\quad (c\mathrm{L} - \mathrm{R},\ d\mathrm{R} - \mathrm{M}),$$

and the three polars,

$$a\mathrm{L} - \mathrm{M},\quad b\mathrm{L} - \mathrm{M},\quad cd\mathrm{L} - 2c\mathrm{R} + \mathrm{M};$$

the three joining lines are

$$b(a+cd)\,\mathrm{L} - 2c(a+b)\,\mathrm{R} + (a+cd)\,\mathrm{M} = 0,$$
$$a(b+cd)\,\mathrm{L} - 2c(a+b)\,\mathrm{R} + (b+cd)\,\mathrm{M} = 0,$$
$$cd\mathrm{L} - \mathrm{M} = 0.$$

Now, the line whose equation we want to construct passes through the intersection of the first of these lines with $b\mathrm{L} - \mathrm{M}$, and of the second with $a\mathrm{L} - \mathrm{M}$.

Ex. 4. Mac Laurin's method of generating conic sections. *The three sides of a triangle pass through three fixed points, and two vertices move on fixed lines, the third vertex will describe a conic section.*

Let the triangle formed by the given points be L, M, N.

Let the given lines be

$$L + aM + bN = 0, \quad (1)$$

$$L + a'M + b'N = 0. \quad (2)$$

Let the base of the triangle be

$$L = \mu M. \quad (3)$$

Substituting this value of L in (1) we find, for the equation of the line joining (1, 3) to (M, N),

$$(\mu + a)\, M + bN = 0.$$

In like manner, the line joining (2, 3) to (L, N) is

$$(\mu + a')\, L + \mu b'N = 0.$$

Eliminating μ from the last two equations, the equation of the locus is

$$a'LM = (aM + bN)\,(L + b'N).$$

The locus is, therefore, a conic passing through the points (L, N), (M, N), (L, 1), (M, 2).

Ex. 5. *The base of a triangle touches a given conic, its extremities move on two fixed tangents to the conic, and the other two sides of the triangle pass through fixed points: find the locus of the vertex.*

Let the fixed tangents be L, M, and the equation of the conic $LM = R^2$.

Let the fixed points be $(aL - R,\ bR - M)$, $(a'L - R,\ b'R - M)$. Then the equation of the line joining $(\mu^2 L - 2\mu R + M,\ L)$ to the first point is

$$a(2\mu - b)\, L - 2\mu R + M = 0.$$

In like manner, the other side of the triangle is

$$\mu a'b'L - 2a'b'R + (2a' - \mu)\, M = 0.$$

Eliminating μ, the locus of the vertex is found to be

$$4a'(aL - R)\,(b'R - M) = (a'b'L - M)\,(abL - M),$$

the equation of a conic through the two given points.

Ex. 6. *If in the last example the extremities of the base lie on any conic having double contact with the given conic, and passing through the given points, to find the locus of the vertex.*

Let the conics be

$$LM - R^2 = 0, \quad LM - \frac{R^2}{\sin^2 2\phi} = 0,$$

then, if any line touch the latter at the point μ, it will, by Art. 275, meet the former in the points $\mu \tan \phi$ and $\mu \cot \phi$, and if the fixed points are μ', μ'', the equations of the sides are

$$\mu\mu' \tan \phi L - (\mu' + \mu \tan \phi) R + M = 0,$$
$$\mu\mu'' \cot \phi L - (\mu'' + \mu \cot \phi) R + M.$$

Eliminating μ, the locus is found to be

$$(M - \mu' R)(\mu'' L - R) = \tan^2 \phi (M - \mu'' R)(\mu' L - R).$$

FOCAL PROPERTIES.

277. The equation $L^2 + M^2 - R^2 = 0$ (p. 215) is one of great importance, and, as well as the equation $LM = R^2$, admits of our expressing the position of any point on the curve by a single indeterminate. We may suppose

$$L = R \cos \phi, \qquad M = R \sin \phi;$$

then, as in Art. 178, the chord joining any two points is

$$L \cos \tfrac{1}{2}(\phi + \phi') + M \sin \tfrac{1}{2}(\phi + \phi') = R \cos \tfrac{1}{2}(\phi - \phi'),$$

and the tangent at any point is

$$L \cos \phi + M \sin \phi = R.$$

The most important application of this equation is in obtaining the properties of the foci. For if $x = 0$, $y = 0$, be any lines at right angles to each other through a focus, and $\gamma = 0$ the equation of the directrix, the equation of the curve is

$$x^2 + y^2 = e^2 \gamma^2,$$

a particular form of the equation we are examining.

The form of the equation shows (Art. 255, XI.) that the focus (xy) is the pole of the directrix γ, and that the polar of any point on the directrix is perpendicular to the line joining it to the focus, for y, the polar of $(x\gamma)$, is perpendicular to x, but x may be any line drawn through the focus.

The form of the equation shows that the two imaginary lines represented by the equation $(x^2 + y^2 = 0)$ are tangents drawn through the focus. Now, since these lines are the same whatever

γ be, it appears that *all conics which have the same focus have two imaginary common tangents passing through this focus.* All conics, therefore, which have *both* foci common, have *four* imaginary common tangents, and may be considered as conics inscribed in the same quadrilateral. The imaginary tangents through the focus ($x^2 + y^2 = 0$) are the same as the lines drawn to the two imaginary points at infinity on any circle (see Art. 253). Hence we obtain the following general conception of foci, which we shall find useful afterwards: "Through each of the two imaginary points at infinity on any circle draw two tangents to the conic; these tangents will form a quadrilateral, two of whose vertices will be real and the foci of the curve, the other two may be considered as imaginary foci of the curve."

278. The tangents through (γ, x) to the curve are evidently $e\gamma + x$ and $e\gamma - x$. If, therefore, the curve be a parabola, $e = 1$; and the tangents are the internal and external bisectors of the angle (γx). Hence, "tangents to a parabola from any point on the directrix are at right angles to each other."

In general, since $x = e\gamma \cos\phi$, $y = e\gamma \sin\phi$, we have

$$\frac{y}{x} = \tan\phi\ ;$$

or ϕ expresses the angle which any radius vector makes with x.

Hence we can find the envelope of a chord which subtends a constant angle at the focus, for the chord

$$x\cos\tfrac{1}{2}(\phi + \phi') + y\sin\tfrac{1}{2}(\phi + \phi') = e\gamma\cos\tfrac{1}{2}(\phi - \phi'),$$

if $\phi - \phi'$ be constant, must, by the present section, always touch

$$x^2 + y^2 = e^2\gamma^2\cos^2\tfrac{1}{2}(\phi - \phi'),$$

a conic having the same focus and directrix as the given one.

279. The line joining the focus to the intersection of two tangents is found by subtracting

$$x\cos\phi + y\sin\phi - e\gamma = 0,$$
$$x\cos\phi' + y\sin\phi' - e\gamma = 0,$$

to be $$x\sin\tfrac{1}{2}(\phi + \phi') - y\cos\tfrac{1}{2}(\phi + \phi') = 0,$$

the equation of a line making an angle $\frac{1}{2}(\phi + \phi')$ with the axis of x, and therefore *bisecting the angle between the focal radii.*

The line joining to the focus the point where the chord of contact meets the directrix is

$$x \cos \tfrac{1}{2}(\phi + \phi') + y \sin \tfrac{1}{2}(\phi + \phi') = 0,$$

a line evidently *at right angles to the last.*

To find the locus of the intersection of tangents at points which subtend a given angle at the focus.

The elimination is precisely the same as in Ex. 4, p. 91, and the locus will be found to be a conic having the same focus and directrix as the given one, and eccentricity $= \dfrac{e}{\cos \frac{1}{2}(\phi - \phi')}$.

If the curve be a parabola, the angle between the tangents is in this case given. For the tangent $(x \cos \phi + y \sin \phi - \gamma)$ bisects the angle between $x \cos \phi + y \sin \phi$ and γ. The angle between the tangents is, therefore, half the angle between $x \cos \phi + y \sin \phi$ and $x \cos \phi' + y \sin \phi'$, or $= \frac{1}{2}(\phi - \phi')$. Hence, *the angle between two tangents to a parabola is half the angle which the points of contact subtend at the focus ;* and again, *the locus of the intersection of tangents to a parabola, which contain a given angle, is a hyperbola with the same focus and directrix, and whose eccentricity is the secant of the given angle,* or whose asymptotes contain double the given angle (Art. 161).

ENVELOPES.

280. We have seen that the line represented by the equation

$$\mu^2 L - 2\mu R + M = 0,$$

always touches the curve $LM = R^2$.

We wish the reader to take notice that this will be the case whether L, M, R represent *right* lines or not. For the equation

$$\mu\mu' L - (\mu + \mu') R + M = 0$$

must be satisfied for any points which satisfy the equations

$$(\mu L - R = 0,\ \mu R - M = 0), \quad (\mu' L - R = 0,\ \mu' R - M = 0),$$

and is therefore the equation of a curve passing through the points in which $\mu L - R$ and $\mu' L - R$ meet $LM - R^2$. Now let $\mu = \mu'$, and we see that $\mu^2 L - 2\mu R + M$ *touches* $LM - R^2$ in the points where $\mu L - R$ meets it.

Similar remarks apply to the equation

$$L\cos\phi + M\sin\phi = R,$$

which indeed may be reduced to the preceding form by assuming $\tan\frac{1}{2}\phi = \mu$, as we have then

$$\cos\phi = \frac{1-\mu^2}{1+\mu^2}, \quad \sin\phi = \frac{2\mu}{1+\mu^2},$$

and substituting these values, and clearing of fractions, we have an equation in which μ only enters in the second degree.

If, therefore, we are required to find the curve always touched by a variable line, we have only to form its equation so as to contain only a single indeterminate, and, if this indeterminate *be only in the second degree*, the envelope can be found as above. We can in like manner find the envelope of a line whose equation contains *two* indeterminates, provided these be connected by some given relation, for we have only to eliminate one of the indeterminates by the help of the given relation.

Ex. 1. *To find the envelope of a line such that the product of the perpendiculars on it from two fixed points may be constant.*

Take for axes the line joining the fixed points and a perpendicular through its middle point, so that the co-ordinates of the fixed points may be $y = 0$, $x = \pm c$; then if the variable line be $y - mx + n = 0$, we have by the conditions of the question

$$(n + mc)\,(n - mc) = b^2(1 + m^2),$$

or
$$n^2 = b^2 + b^2m^2 + c^2m^2,$$

but
$$n^2 = y^2 - 2mxy + m^2x^2,$$

therefore
$$m^2(x^2 - b^2 - c^2) - 2mxy + y^2 - b^2 = 0;$$

and the envelope is

$$x^2y^2 = (x^2 - b^2 - c^2)\,(y^2 - b^2),$$

or
$$\frac{x^2}{b^2 + c^2} + \frac{y^2}{b^2} = 1.$$

Ex. 2. *Through a fixed point* O *any line* OP *is drawn to meet a fixed line; to find the envelope of* PQ *drawn so as to make the angle* OPQ *constant.*

Let OP make the angle θ with the perpendicular on the fixed line, and its length is $p\sec\theta$; but the perpendicular from O on

PQ makes a fixed angle β with OP, therefore its length is $= p \sec\theta \cos\beta$; and since this perpendicular makes an angle $= \theta + \beta$ with the perpendicular on the fixed line, if we assume the latter for the axis of x, the equation of PQ is

$$x\cos(\theta+\beta) + y\sin(\theta+\beta) = p\sec\theta\cos\beta,$$

or $$x\cos(2\theta+\beta) + y\sin(2\theta+\beta) = 2p\cos\beta - x\cos\beta - y\sin\beta,$$

an equation of the form

$$\mathrm{L}\cos\phi + \mathrm{M}\sin\phi = \mathrm{R},$$

whose envelope, therefore, is

$$x^2 + y^2 = (x\cos\beta + y\sin\beta - 2p\cos\beta)^2,$$

the equation of a parabola having the point O for its focus.

Ex. 3. *To find the envelope of the line* $\frac{\mathrm{A}}{\mu} + \frac{\mathrm{B}}{\mu'} = 1$, *where the indeterminates are connected by the relation* $\mu + \mu' = \mathrm{C}$.

We may substitute for μ', $\mathrm{C} - \mu$, and clear of fractions; the envelope is thus found to be

$$\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 - 2\mathrm{AB} - 2\mathrm{AC} - 2\mathrm{BC} = 0,$$

an equation to which the following form will be found to be equivalent,

$$\pm\sqrt{\mathrm{A}} \pm \sqrt{\mathrm{B}} \pm \sqrt{\mathrm{C}} = 0.$$

Thus, for example, *Given vertical angle and sum of sides of a triangle, to find the envelope of base.*

The equation of the base is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

where $a + b = c$.

The envelope is, therefore,

$$x^2 + y^2 - 2xy - 2cx - 2cy + c^2 = 0,$$

a parabola touching the sides x and y.

In like manner, *Given in position two conjugate diameters of an ellipse, and the sum of their squares, to find its envelope.*

If in the equation

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$$

we have $a'^2 + b'^2 = c^2$, the envelope is

$$x \pm y \pm c = 0.$$

The ellipse, therefore, must always touch four fixed right lines.

Ex. 4. Again,* given the two equations

$$\frac{\mu}{A} + \frac{\mu'}{B} + \frac{\mu''}{C} = 0,$$

$$\sqrt{(\mu a)} + \sqrt{(\mu' b)} + \sqrt{(\mu'' c)} = 0,$$

if we eliminate μ'', the equation in $\frac{\mu}{\mu'}$ will be only of the second order, and the envelope will be found to be

$$Aa + B\beta + C\gamma = 0.\dagger$$

Thus, for example, in the equation of a conic circumscribing a triangle,

$$\frac{\mu}{\alpha} + \frac{\mu'}{\beta} + \frac{\mu''}{\gamma} = 0$$

(Art. 102), if the constants be connected by the relation

$$\sqrt{(\mu a)} + \sqrt{(\mu' b)} + \sqrt{(\mu'' c)} = 0,$$

the conic will touch the right line

$$a\alpha + b\beta + c\gamma = 0.$$

Or, again, in the equation of a conic inscribed in a triangle,

$$\sqrt{(\mu\alpha)} + \sqrt{(\mu'\beta)} + \sqrt{(\mu''\gamma)} = 0$$

(Art. 105), if the constants be connected by the relation

$$\frac{\mu}{A} + \frac{\mu'}{B} + \frac{\mu''}{C} = 0,$$

the conic will touch the right line

$$A\alpha + B\beta + C\gamma = 0.$$

281. These principles enable us to write the equation of a conic having double contact with two given conics, S and S'. Let

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* This example, and its applications, are taken from Mr. Hearn's *Researches on Conic Sections.*

† In general, given the two equations

$$(\mu A)^m + (\mu' B)^m + (\mu'' C)^m = 0,$$
$$(\mu a)^n + (\mu' b)^n + (\mu'' c)^n = 0,$$

it can be proved that the envelope is

$$\left(\frac{a}{A}\right)^{\frac{mn}{m-n}} + \left(\frac{b}{B}\right)^{\frac{mn}{m-n}} + \left(\frac{c}{C}\right)^{\frac{mn}{m-n}} = 0.$$

E and F be their chords of intersection, so that $S - S' = EF$, then the equation of any conic touching the two will be

$$\mu^2 E^2 - 2\mu(S + S') + F^2 = 0.$$

For, if we seek the envelope of this conic, we find

$$E^2F^2 - (S + S')^2 = 0, \text{ or } 4SS' = 0;$$

hence this conic touches both the given ones.

Since μ is of the second degree, we see that through any point can be drawn *two* conics, each of which will have double contact with the given ones; and it can be proved that one of the chords of intersection of these conics is the line joining the given point to (EF), and the other the fourth harmonic to this line, E and F.

The equation of a conic inscribed in a *quadrilateral* is a particular case of the foregoing, and is

$$\mu^2 E^2 - 2\mu(AC + BD) + F^2 = 0,$$

where ABCD are the sides, EF the diagonals, and $AC - BD = EF$.

282. The equation of a conic having double contact with two *circles* can be expressed in a simpler form, viz.,

$$\mu^2 - 2\mu(C + C') + (C - C')^2 = 0.$$

The chords of contact of the conic with the circles are found to be

$$C - C' + \mu = 0, \text{ and } C - C' - \mu = 0,$$

which are, therefore, parallel to each other, and equidistant from the radical axis of the circles. This equation may also be written in the form

$$\sqrt{C} \pm \sqrt{C'} = \sqrt{\mu}.$$

Hence, *the locus of a point, the sum or difference of whose tangents to two given circles is constant, is a conic having double contact with the two circles.* If we suppose both circles infinitely small, we obtain the fundamental property of the foci of the conic.

GENERAL EQUATION OF THE SECOND DEGREE.

283. We have already seen that the general trilinear equation of the second degree is

$$A\alpha^2 + B\alpha\beta + C\beta^2 + D\alpha\gamma + E\beta\gamma + F\gamma^2 = 0.$$

We may reduce the number of *expressed* constants by including

three of them implicitly in the equations of the lines of reference, and writing $\alpha\sqrt{A} = L, \quad \beta\sqrt{C} = M, \quad \gamma\sqrt{F} = N,$

and the equation of the conic may be written in the form

$$L^2 + M^2 + N^2 + 2nLM + 2lMN + 2mNL = 0.$$

This equation is evidently equivalent to the equation

$$(L + nM + mN)^2 + (1 - n^2)\,M^2 + (1 - m^2)\,N^2 + 2(l - mn)\,MN;$$

but the last three terms are the equation of two right lines drawn through (MN); hence (Art. 255, X.) $L + nM + mN$ is the chord of contact of two tangents drawn through (MN), that is to say, the *polar* of the point (MN).

In like manner, the polar of NL is

$$M + lN + nL = 0,$$

and the polar of LM is

$$N + lM + nL = 0.$$

It can hence readily be inferred that *the three points in which any triangle meets the sides of its polar triangle are in one right line,*

$$\frac{L}{l} + \frac{M}{m} + \frac{N}{n} = 0.$$

Again, *the lines joining the vertices of any triangle to the corresponding vertices of the polar triangle will meet in a point* (see page 96). For these lines can easily be proved to be

$$(l - mn)\,L - (m - nl)\,M = 0,$$
$$(m - nl)\,M - (n - lm)\,N = 0,$$
$$(n - lm)\,N - (l - mn)\,L = 0.$$

Want of space compels us to omit some examples of questions solved by means of the general trilinear equation. This equation is, in general, less convenient than the form ($LM = R^2$), because we get a quadratic to determine the points of intersection with the curve of any line through (LM). The student may examine by this method (Ex. 3, p. 234), and he will find that the solution, although more troublesome to obtain than the solution there given, yet leads more easily to the geometrical construction.

284. The form of the equation of the tangents through (LM) leads to an important property of the sides of a circumscribing

hexagon, and affords a useful test for determining whether six lines touch a conic.

The tangents are

$$(1 - m^2)\,L^2 + 2\,(n - lm)\,LM + (1 - l^2)\,M^2 = 0,$$
$$(1 - n^2)\,M^2 + 2\,(l - mn)\,MN + (1 - m^2)N^2 = 0,$$
$$(1 - l^2)\,N^2 + 2\,(m - ln)\,NL + (1 - n^2)\,L^2 = 0.$$

Now, if the roots of the first equation be $L = \mu M$, $L = \mu' M$, we have

$$\mu\mu' = \frac{1 - l^2}{1 - m^2}.$$

The corresponding quantities for the other equations are $\frac{1 - n^2}{1 - l^2}$ and $\frac{1 - m^2}{1 - n^2}$, and these three multiplied together are = 1. Now, recollecting the meaning of μ (Art. 53) we learn, that if A, B, C, D, E, F, be the vertices of a circumscribing hexagon,

$$\frac{\sin EAB \cdot \sin FAB \cdot \sin FBC \cdot \sin DBC \cdot \sin DCA \cdot \sin ECA}{\sin EAC \cdot \sin FAC \cdot \sin FBA \cdot \sin DBA \cdot \sin DCB \cdot \sin ECB} = 1.$$

Hence, also, if the equations of three pairs of lines can be put into the form

$$L^2 + M^2 - 2n'LM = 0,$$
$$M^2 + N^2 - 2l'MN = 0,$$
$$N^2 + L^2 - 2m'NL = 0,$$

they will touch the same conic section, for the equations last given can be reduced to this form by writing $\sqrt{(1 - l^2)}\,L$ for L, &c.

285. We saw that the polar of any point $(\beta\gamma)$, with regard to the conic, $S = 0$, or $\alpha^2 + 2\alpha(m\beta + n\gamma) + \beta^2 + \gamma^2 + 2l\beta\gamma = 0$, is

$$\alpha + m\beta + n\gamma = 0.$$

Now, this is the *first derived equation* of $S = 0$, *considered as a function of* α. We shall anticipate the notation of the calculus, and denote this derived equation by $\frac{dS}{d\alpha}$.

In like manner, the polar of $(\alpha\gamma)$, with regard to S, is the first derived equation of S, considered as a function of β, $= \frac{dS}{d\beta}$, and the polar of $\alpha\beta$ is $\frac{dS}{d\gamma}$. Hence, *if the equation of a conic be expressed in terms of the equations of three right lines, the equation of the polar of the intersection of any two of them is the first derived*

of the equation of the conic, considered as a function of the third line. The equations of polars given already are particular cases of this. For example, the polar of the origin (xy), with regard to

$$Ax^2 + Bxy + Cy^2 + Dxk + Eyk + Fk^2 = 0,$$

is

$$Dx + Ey + 2Fk = 0,$$

that is, its first derived equation with regard to k.

Again, the equation of the diameter which bisects chords parallel to the axis of x is

$$\frac{dS}{dx} = 0, \quad \text{or } 2Ax + By + Dk = 0,$$

and we shall show hereafter that this diameter may be considered as the polar of the point (yk) at infinity on the axis of x.

286. Ex. 1. *Given four points on a conic, the polar of any other given point will pass through a fixed point.*

The equation of the conic must be of the form $S - kS' = 0$, where S and S′ are any two conics through the four points: now the polar of any point $\beta\gamma$ with regard to this is $\frac{d(S - kS')}{d\alpha} = 0$, which, it will be seen, is equivalent to

$$\frac{dS}{d\alpha} - k\frac{dS'}{d\alpha} = 0;^*$$

and since this equation only involves k in the first degree, it will pass through a fixed point.

Ex. 2. *To find the locus of the pole of a given line* (γ), *with regard to a conic of which four points are given.*

We have to eliminate k from the equations

$$\frac{dS}{d\alpha} - k\frac{dS'}{d\alpha} = 0,$$

$$\frac{dS}{d\beta} - k\frac{dS'}{d\beta} = 0,$$

* We may mention here, that if the axes of S be parallel to the axes of S′, so will the axes of $S - kS'$; for if we take the axes of S for axes of co-ordinates, neither S nor S will contain the term xy. If S′ be a circle, the axes of $S - kS'$ must be always parallel to the axes of S. If $S - kS'$ reduce to a pair of right lines, its axes will become the internal and external bisectors of the angles between these right lines: thus we obtain the theorem of p. 208.

and we find $$\frac{dS}{da}\frac{dS'}{d\beta} - \frac{dS}{d\beta}\frac{dS'}{da} = 0,$$

the equation of a conic section.

If we suppose the given line at an infinite distance, we obtain the locus of the centres (Art. 152).

Ex. 3. *Given two points and two tangents to a conic, the polar of a fixed point touches a conic section.*

Let LM be the two tangents, R the line joining the given points, and LM − N^2 one conic touching the two lines, and passing through the given points; then the equation of any other must be of the form

$$LM - (N + kR)^2 = 0;$$

the polar is therefore

$$2k^2R\frac{dR}{da} + 2k.\frac{d(NR)}{da} + \frac{d(N^2 - LM)}{da} = 0,$$

which must always touch a conic section, since k enters in the second degree. In the same manner it may be proved that the locus of the pole of a given line is a conic section.

In general, if the equation of a conic section involve an indeterminate in the second degree, the polar of any fixed point will touch a conic section. Thus, for example, the locus of centres of conic sections which have double contact with two given conics (Art. 281) is a conic section.

287. *To find the equation of the polar of any point* $(\alpha'\beta'\gamma')$, *with regard to a conic section.*

This may be done by a method similar to that used Art. 145. For $l\alpha'' + m\alpha'$, $l\beta'' + m\beta'$, $l\gamma'' + n\gamma'$, are the trilinear co-ordinates of a point somewhere on the line joining $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ (see note, p. 57), since these values will satisfy any equation $A\alpha + B\beta + C\gamma = 0$, which is satisfied for both these points. If then we substitute these values in the general equation $S = 0$, we have, to determine the points where this conic is met by the line joining $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$, the quadratic

$$\begin{aligned} &l^2(A\alpha''^2 + B\alpha''\beta'' + C\beta''^2 + D\alpha''\gamma'' + E\beta''\gamma'' + F\gamma''^2) \\ &+ lm\{(2A\alpha'' + B\beta'' + D\gamma'')\,\alpha' + (B\alpha'' + 2C\beta'' + E\gamma'')\,\beta' \\ &\qquad + (D\alpha'' + E\beta'' + 2F\gamma'')\,\gamma'\} \\ &+ m^2(A\alpha'^2 + B\alpha'\beta' + C\beta'^2 + D\alpha'\gamma' + E\beta'\gamma' + F\gamma'^2) = 0. \end{aligned}$$

Now, as in Art. 145, when $\alpha''\beta''\gamma''$ is on the polar of $\alpha'\beta'\gamma'$, the coefficient of lm must vanish, since we know that the line joining the points must in this case be cut harmonically; the equation of the polar of $\alpha'\beta'\gamma'$ is, therefore,

$$(2A\alpha + B\beta + D\gamma)\alpha' + (B\alpha + 2C\beta + E\gamma)\beta' + (D\alpha + E\beta + 2F\gamma)\gamma' = 0,$$

which we may write for shortness

$$\alpha' \frac{dS}{d\alpha} + \beta' \frac{dS}{d\beta} + \gamma' \frac{dS}{d\gamma} = 0.$$

Hence if the three lines $\frac{dS}{d\alpha}, \frac{dS}{d\beta}, \frac{dS}{d\gamma}$ meet in a point, the polar of any other point will pass through the same point. This will be the case when the conic resolves itself into two right lines, for then the polar of any point is a fourth harmonic through the intersection of the two lines to the line joining this intersection to the given point.

If the conic be

$$L^2 + M^2 + N^2 - 2lMN - 2mNL - 2nLM = 0,$$

the condition that

$$L - nM - mN, \quad M - lN - nL, \quad N - mL - lM,$$

should meet in a point, is proved by eliminating $\frac{M}{N}, \frac{L}{N}$, to be

$$1 - l^2 - m^2 - n^2 - 2lmn = 0.$$

If l, m, n are less than 1, we may put $l = \cos\theta$, $m = \cos\theta'$, $n = \cos\theta''$, and the condition takes the simple form,

$$\theta + \theta' + \theta'' = 180°.$$

INSCRIBED AND CIRCUMSCRIBED TRIANGLES.

288. We gave (p. 102) the equation of a conic circumscribed about a triangle,*

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$

we may prove, precisely as at p. 104, that the tangents at the three vertices are

$$l\beta + m\alpha = 0, \quad m\gamma + n\beta = 0, \quad n\alpha + l\gamma = 0;$$

* This equation was, I believe, first discussed by M. Bobillier (*Annales de Mathématiques*, vol. xviii. p. 320).

that the three points in which each tangent meets the opposite side are in one right line,

$$\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n} = 0;$$

and that the lines joining each vertex to the opposite vertex of the circumscribed triangle are

$$\frac{\alpha}{l} - \frac{\beta}{m} = 0, \quad \frac{\beta}{m} - \frac{\gamma}{n} = 0, \quad \frac{\gamma}{n} - \frac{\alpha}{l} = 0,$$

which evidently meet in a point.

To find the equation of a conic circumscribing $\alpha\beta\gamma$, *and having its centre at a given point* $(\alpha'\beta'\gamma')$.

The polar of any point is (Art. 287)

$$\alpha'(m\gamma + n\beta) + \beta'(n\alpha + l\gamma) + \gamma'(l\beta + m\alpha) = 0.$$

Now it is required to determine lmn, so that this equation should represent a line at an infinite distance (Art. 152).

Comparing this equation, therefore, with the equation of a line at infinity (Art. 64),

$$a\alpha + b\beta + c\gamma = 0,$$

where abc are the lengths of the sides of the triangle $\alpha\beta\gamma$, it will be found that we may take

$l = \alpha'(b\beta' + c\gamma' - a\alpha')$; $m = \beta'(a\alpha' + c\gamma' - b\beta')$; $n = \gamma'(a\alpha' + b\beta' - c\gamma')$.*

In like manner we could determine l, m, n, so that the polar of $(\alpha'\beta'\gamma')$ should be any right line, $A\alpha + B\beta + C\gamma$, by writing A, B, C, for a, b, c.

If we were given three points on a conic and any fourth condition, this fourth condition will give a relation between l, m, n; then by writing in this relation the values of l, m, n, just found, we can find the locus of centres of the conic, or the locus of the poles of a given line.† Thus, for example, if we are given a fourth point on the conic, we must have

$$\frac{l}{\alpha''} + \frac{m}{\beta''} + \frac{n}{\gamma''} = 0,$$

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* $a\alpha + b\beta - c\gamma$ is the line joining the middle points of $\alpha\beta$. I do not perceive, however, that these values admit of any simple geometrical interpretation.

† The method given in this and the following Article, of finding the locus of the centre of a conic section described under certain conditions, is taken from Mr. Hearn's *Researches on Conic Sections*.

and therefore the locus of the centre of the conic circumscribing a quadrilateral is

$$\frac{a(b\beta + c\gamma - a\alpha)}{\alpha''} + \frac{\beta(a\alpha + c\gamma - b\beta)}{\beta''} + \frac{\gamma(a\alpha + b\beta - c\gamma)}{\gamma''} = 0,$$

a conic through the middle points of the given quadrilateral.

If we are given a tangent to the conic we must have

$$\sqrt{(l\mathrm{A})} + \sqrt{(m\mathrm{B})} + \sqrt{(n\mathrm{C})} = 0,$$

in order that the conic should touch

$$\mathrm{A}\alpha + \mathrm{B}\beta + \mathrm{C}\gamma = 0 \text{ (Art. 280)},$$

therefore the locus of centre, three points and a tangent being given, is

$$\sqrt{\{\mathrm{A}\alpha(b\beta + c\gamma - a\alpha)\}} + \sqrt{\{\mathrm{B}\beta(a\alpha + c\gamma - b\beta)\}} + \sqrt{\{\mathrm{C}\gamma(b\beta + a\alpha - c\gamma)\}} = 0,$$

a curve in general of the fourth degree.

289. The equation of the conic section *inscribed* in a triangle may be written in either of the forms (Art. 105)

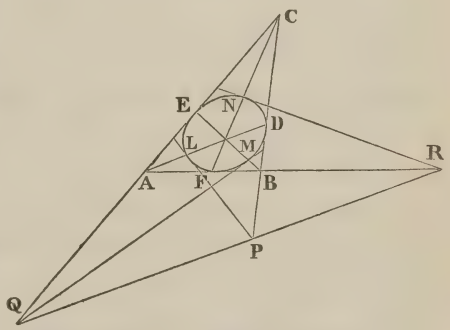
$$\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0,$$

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0.$$

It was proved (Art. 106) that AD, BE, CF meet in a point, their equations being

$$m\beta - n\gamma = 0, \quad n\gamma - l\alpha = 0, \quad l\alpha - m\beta = 0;$$

that LP, MQ, NR have for their equations respectively,



$$2m\beta + 2n\gamma - l\alpha = 0, \quad 2n\gamma + 2l\alpha - m\beta = 0, \quad 2l\alpha + 2m\beta - n\gamma = 0,$$

and that PQR is a right line whose equation is

$$l\alpha + m\beta + n\gamma = 0.$$

It is evident likewise that CA, CF, CB, CR form a harmonic pencil, their equations being

$$\beta = 0, \quad l\alpha - m\beta = 0, \quad \alpha = 0, \quad l\alpha + m\beta = 0.$$

To find the equation of a conic inscribed in $a\beta\gamma$, and having its centre at a given point $(a'\beta'\gamma')$.

The polar of any point with regard to this conic is (Art. 287)

$$al(m\beta' + n\gamma' - la') + \beta m(la' + n\gamma' - m\beta') + \gamma n(la' + m\beta' - n\gamma') = 0.$$

Now if it were required to determine l, m, n, so that this polar should coincide with

$$\mathrm{L}a + \mathrm{M}\beta + \mathrm{N}\gamma = 0,$$

we should find

$$l = \mathrm{L}(\mathrm{M}\beta' + \mathrm{N}\gamma' - \mathrm{L}a'); \quad m = \mathrm{M}(\mathrm{L}a' + \mathrm{N}\gamma' - \mathrm{M}\beta');$$
$$n = \mathrm{N}(\mathrm{L}a' + \mathrm{M}\beta' - \mathrm{N}\gamma').$$

Hence the locus of the centres of a conic touching three lines, and passing through a given point $a''\beta''\gamma''$, is

$$\sqrt{\{aa''(b\beta + c\gamma - aa)\}} + \sqrt{\{b\beta''(aa + c\gamma - b\beta)\}} + \sqrt{\{c\gamma''(aa + b\beta - c\gamma)\}} = 0,$$

the equation of a conic touching the lines joining the middle points of the sides of the triangle formed by the given tangents.

If the conic touch a fourth given line, $\mathrm{A}a + \mathrm{B}\beta + \mathrm{C}\gamma = 0$, we must (Art. 280) have the relation

$$\frac{l}{\mathrm{A}} + \frac{m}{\mathrm{B}} + \frac{n}{\mathrm{C}} = 0;$$

the locus of the centre is, therefore,

$$\frac{a(b\beta + c\gamma - aa)}{\mathrm{A}} + \frac{b(aa + c\gamma - b\beta)}{\mathrm{B}} + \frac{c(aa + b\beta - c\gamma)}{\mathrm{C}} = 0,$$

the equation of a right line.*

Thus too we may easily form the equation of a conic touching five given right lines, viz., a, β, γ, $\mathrm{A}a + \mathrm{B}\beta + \mathrm{C}\gamma$, $\mathrm{A}'a + \mathrm{B}'\beta + \mathrm{C}'\gamma$; for we have the two equations

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* The condition that a conic circumscribed about the triangle $(a\beta\gamma)$,

$$\frac{l}{a} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$

should touch another inscribed in it,

$$\sqrt{(\mathrm{L}a)} + \sqrt{(\mathrm{M}\beta)} + \sqrt{(\mathrm{N}\gamma)} = 0,$$

is (note, p. 241) $(l\mathrm{L})^{\frac{1}{3}} + (m\mathrm{M})^{\frac{1}{3}} + (n\mathrm{N})^{\frac{1}{3}} = 0$;

hence we can find the locus of the centre of the conic inscribed in a given triangle, and touching another circumscribed to the same triangle, or *vice versâ* (Hearn, p. 50).

$$\frac{l}{A} + \frac{m}{B} + \frac{n}{C} = 0, \qquad \frac{l}{A'} + \frac{m}{B'} + \frac{n}{C'} = 0,$$

from which we can determine $l : m$ and $l : n$.

CHAPTER XIV.

GEOMETRICAL METHODS.

290. In treating of curves of higher dimensions, we feel it necessary to adopt a method different from that which we have used in the case of the right line and circle. In the first part of this work we supposed the reader already acquainted with the properties of the figures there treated of, and our object was to teach, not the properties themselves, but the analytic method of investigating them. In our future chapters, on the contrary, we may suppose the reader, although unacquainted with the properties of the higher curves, yet to be tolerably familiar with the practice of the method of co-ordinates; we shall, therefore, make *the theory of curves* our primary object; and, without confining ourselves to the exclusive use of algebraic methods, shall employ, in each particular case, whatever mode of investigation appears best suited to the nature of the subject we are considering. We purpose to occupy the present chapter with some important geometrical methods, an account of which must form an essential part of any work devoted to the theory of curves.

THE METHOD OF RECIPROCAL POLARS.*

291. Being given a fixed conic section (Σ) and any curve (S), we can generate another curve (s) as follows: draw any tangent to S, and take its pole with regard to Σ, the locus of this pole will be a curve s, which is called the *polar curve* of S with regard to Σ. The conic Σ, with regard to which the pole is taken, is called the *auxiliary* conic.

* This beautiful method was introduced by M. Poncelet, whose account of it will be found at the commencement of the fourth volume of Crelle's Journal.

We have already met with a particular example of polar curves (Art. 235), where we proved that the polar curve of a conic section,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (S),$$

with regard to another conic section,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad (\Sigma),$$

is always a curve of the second degree.

We shall for brevity say that a point *corresponds* to a line when we mean that the point is the pole of that line with regard to Σ; thus, since it appears from our definition that every point of s is the pole with regard to Σ of some tangent to S, we shall briefly express this relation by saying that every point of s *corresponds* to some tangent of S.

292. *The point of intersection of two tangents to* S *will correspond to the line joining the corresponding points of* s.

This follows from the property of the conic Σ, that the point of intersection of any two lines is the pole of the line joining the poles of these two lines (Art. 141).

Let us suppose that in this theorem the two tangents to S are indefinitely near, then the two corresponding points of s will also be indefinitely near, and the line joining them will be a tangent to s (Art. 83); it also easily follows, from our definition of a tangent, that any tangent to a curve intersects the consecutive tangent at its point of contact (see Art. 130); hence for this case the last theorem becomes: *If any tangent to* S *correspond to a point on* s, *the point of contact of that tangent to* S *will correspond to the tangent through the point on* s.

Hence we see that the relation between the curves is *reciprocal*, that is to say, that the curve S might be generated from s in precisely the same manner that s was generated from S; hence the name "reciprocal polars."

293. We are now able, being given any theorem *of position* (Art. 1) concerning any curve S, to deduce another concerning the curve s. Thus, for example, if we know that a number of points connected with the figure S lie on one right line, we learn that the corresponding lines connected with the figure s meet in

a point (Art. 141), and *vice versâ ;* if a number of points connected with the figure S lie on a conic section, the corresponding lines connected with s will touch the polar of that conic with regard to Σ, or, in general, if the *locus* of any point connected with S be any curve S′, the *envelope* of the corresponding line connected with s is s', the reciprocal polar of S′.

The degree of the polar reciprocal of any curve is equal to the number of tangents which can be drawn from any point to that curve.

For the degree of s is the same as the number of points in which any line cuts s, and to a number of points on s, lying on a right line, correspond *the same number* of tangents to S passing through the point corresponding to that line. Thus, for example, if S be a conic section, two, and only two, tangents, real or imaginary, can be drawn to it from any point (Art. 130); therefore, any line meets s in two, and only two points, real or imaginary; we may thus infer (see note, p. 209), independently of Art. 235, that the reciprocal of any conic section is a curve of the second degree.

294. We shall exemplify, in the case where S and s are conic sections, the mode of obtaining one theorem from another by this method.

We know (Art. 266) that "if a hexagon be *in*scribed in S, whose *sides* are A, B, C, D, E, F, then the *points* of intersection, AD, BE, CF, are *in one right line*." Hence we infer, that "if a hexagon be *circum*scribed about s, whose *vertices* are a, b, c, d, e, f, then the *lines* ad, be, cf, will *meet in a point*" (Art. 264). Thus we see that Pascal's theorem and Brianchon's are reciprocal to each other, and it was thus, in fact, that the latter was first obtained.

In order to give the student an opportunity of rendering himself expert in the application of this method, we shall write in parallel columns some theorems, together with their reciprocals. The beginner ought carefully to examine the force of the argument by which the one is inferred from the other, and he ought to attempt to form for himself the reciprocal of each theorem before looking at the reciprocal we have given. He will soon find that the operation of forming the reciprocal theorem will reduce

itself to a mere mechanical process of interchanging the words "point" and "line," "inscribed" and "circumscribed," "locus" and "envelope," &c.

If two vertices of a triangle move along fixed right lines, while the sides pass each through a fixed point, the locus of the third vertex is a conic section. (Art. 276, Ex. 4.)

If, however, the points through which the sides pass lie in one right line, the locus will be a right line. (p. 42.)

In what other case will the locus be a right line? (p. 44.)

If two sides of a triangle pass through fixed points, while the vertices move on fixed right lines, the envelope of the third side is a conic section.

If the lines on which the vertices move meet in a point, the third side will pass through a fixed point. (p. 61.)

In what other case will the third side pass through a fixed point? (p. 62.)

If two conics touch, their reciprocals will also touch; for the first pair have a point common, and also the tangent at that point common, therefore the second pair will have a tangent common and its point of contact also common. So likewise if two conics have double contact their reciprocals will have double contact.

If a triangle be circumscribed to a conic section, two of whose vertices move on fixed lines, the locus of the third vertex is a conic section, having double contact with the given one. (Art. 276, Ex. 1.)

If a triangle be inscribed in a conic section, two of whose sides pass through fixed points, the envelope of the third side is a conic section, having double contact with the given one. (Art. 276, Ex. 2.)

295. We proved (Art. 292, see figure, p. 257) if to two points P, P′, on S, correspond the tangents pt, $p't'$, on s, that the tangents at P and P′ will correspond to the points of contact p, p', and therefore Q, the intersection of these tangents, will correspond to the chord of contact pp'. Hence we learn that *to any point* Q, *and its polar* PP′, *with respect to* S, *correspond a line* pp' *and its pole* q *with respect to* s.

Given two points on a conic, and two of its tangents, the line joining the points of contact of those tangents passes through a fixed point. (Art. 263.)

Given four points on a conic, the

Given two tangents and two points on a conic, the point of intersection of the tangents at those points will move along a fixed right line.

Given four tangents to a conic,

polar of a fixed point passes through a fixed point. (Art. 236, Ex. 4.)

Given four points on a conic, the locus of the pole of a fixed right line is a conic section. (Art. 286.)

The lines joining the vertices of a triangle to the opposite vertices of its polar triangle with regard to a conic, meet in a point. (Art. 283.)

Inscribe in a conic a triangle whose sides pass through three given points. (p. 234.)

the locus of the pole of a fixed right line is a right line.

Given four tangents to a conic, the envelope of the polar of a fixed point is a conic section.

The points of intersection of each side of any triangle, with the opposite side of the polar triangle, lie in one right line.

Circumscribe about a conic a triangle whose vertices rest on three given lines.

296. Given two conics, S and S′, and their two reciprocals, s and s'; to any point common to S and S′ will correspond a tangent common to s and s', and to any chord of intersection of S and S′ will correspond an intersection of common tangents to s and s'.

If three conics have two points common, and, therefore, one common chord, their other three common chords will meet in a point. (Art. 265.)

If three conics have two common tangents, or if they have each double contact with a fourth, their six chords of intersection will pass three by three through the same points (Art. 264.)

Or, in other words, three conics, having each double contact with a fourth, may be considered as having four radical centres. (p. 108.)

If three conics have two tangents common, the points of intersection of the other three pairs of common tangents lie on one right line.

If three conics have two points common, or if they have each double contact with a fourth, the six points of intersection of common tangents lie three by three on the same right lines.

Or, in other words, three conics, having each double contact with a fourth, may be considered as having four axes of similitude. (See Art. 118, of which this theorem is an extension.)

If through the point of contact of two conics which touch, any chord be drawn, tangents at its extremities will meet on the common chord of the two conics.

If from any point on the tangent at the point of contact of two conics which touch, a tangent be drawn to each, the line joining their points of contact will pass through the intersection of common tangents to the conics.

If, through the intersection of common tangents of two conics any two chords be drawn, lines joining their extremities will intersect on one or

If, on a common chord of two conics, any two points be taken, and from these tangents be drawn to the conics, the diagonals of the quadrila-

other of the common chords of the two conics. (p. 221.)

If A and B be two conics having each double contact with S, the chords of contact of A and B with S, and their chords of intersection with each other, meet in a point and form a harmonic pencil. (p. 219.)

If A, B, C, be three conics, having each double contact with S, and if A and B both touch C, the tangents at the points of contact will intersect on a common chord of A and B.

teral so formed will pass through one or other of the intersections of common tangents to the conics.

If A and B be two conics having each double contact with S, the intersections of the tangents at their points of contact with S, and the intersections of tangents common to A and B, lie in one right line, which they divide harmonically.

If A, B, C, be three conics, having each double contact with S, and if A and B both touch C, the line joining the points of contact will pass through an intersection of common tangents of A and B.*

297. We have hitherto supposed the auxiliary conic Σ to be any conic whatever. It is most common, however, to suppose this conic a circle; and hereafter, when we speak of polar curves, we intend the reader to understand polars *with regard to a circle*, unless we expressly state otherwise.

We know (Art. 89) that the polar of any point with regard to a circle is perpendicular to the line joining this point to the

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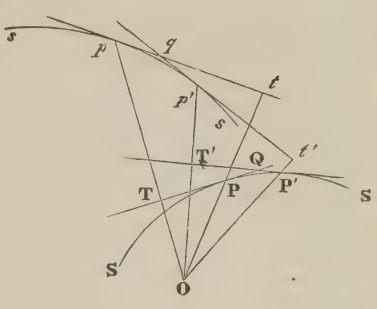
* The reader will take notice that we have now proved that every theorem used in Art. 123 in the theory of three circles has a theorem corresponding in the theory of three conics which are each inscribed in the same given conic; and hence that, given three such conics, we can find a fourth inscribed in the same conic, and such as to touch the three given conics. The learner will do well to refer to Art. 123, and to examine for himself how the demonstration there given is to be extended to the case of three conics inscribed in a given conic. The chief difference occurs in (5) of that Article, for the line $a''b''$ is now constructed by joining the pole of SS'S'' to *any one of the four* radical centres of the three conics. The problem therefore admits of thirty-two solutions instead of eight, as in the case of the three circles. The theorems which answer to (6) of the same Article are the following:

The chord of contact of the required conic with S passes through the intersection of one of the axes of similitude of the three given conics with the polar of one of their radical centres with regard to S.

The pole of this chord, with regard to S, lies on the line joining one of their radical centres with the pole, with regard to S, of one of their axes of similitude.

The reader will find a very able investigation of this whole problem in a memoir published by Mr. Cayley in vol. 39 of Crelle's Journal.

centre, and that the distances of the point and its polar are, when multiplied together, equal to the square of the radius; hence the relation between polar curves with regard to a circle is often stated as follows: *Being given any point* O, *if from it we let fall a perpendicular* OT *on any tangent to a curve* S, *and produce it until the rectangle* OT.O*p is equal to a constant* k^2, *then the locus of the point p is a curve s, which is called the polar reciprocal of* S. For this is evidently equivalent to saying that p is the pole of PT, with regard to a circle whose centre is O and radius k. We see, therefore (Art. 292), that the tangent pt will correspond to the point of contact P, that is to say, that OP will be perpendicular to pt, and that $OP.Ot = k^2$.



It is easy to show that a change in the magnitude of k will affect only the *size* and not the *shape* of s, which is all that in most cases concern us. In this manner of considering polars all mention of the circle may be suppressed, and s may be called the reciprocal of S *with regard to the point* O. We shall call this point the *origin*.

The *advantage* of using the circle for our auxiliary conic chiefly arises from the two following theorems, which are at once deduced from what has been said, and which enable us to transform, by this method, not only theorems of position, but also theorems involving the magnitude of lines and angles:

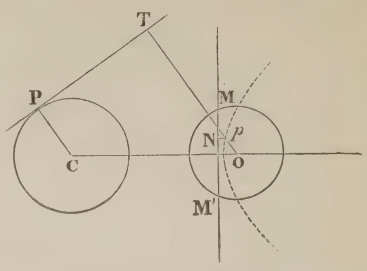
The distance of any point P *from the origin is the reciprocal of the distance of the corresponding line pt.*

The angle TQT′ *between any two lines* TQ, T′Q, *is equal to the angle* pOp' *subtended at the origin by the corresponding points* p, p', for Op is perpendicular to TQ, and Op' to T′Q.

We shall give some examples of the application of these principles when we have first investigated the following problem:

298. *To find the polar reciprocal of one circle with regard to another.* That is to say, to find the locus of the pole p with re-

gard to the circle (O) of any tangent PT to the circle (C). Let MN be the polar of the point C with regard to O, then having the points C, p, and their polars MN, PT, we have by Ex. 3, p. 95, the ratio $\frac{OC}{CP} = \frac{Op}{pN}$, but the first ratio is constant, since both OC and CP are constant; hence the distance of p from O is to its distance from MN in the constant ratio $\frac{OC}{CP}$, its locus is therefore a conic, of which O is a focus, MN the corresponding directrix, and $\frac{OC}{CP}$ the eccentricity. Hence the eccentricity is greater, less than, or = 1, according as O is without, within, or on the circle C.



Hence the *polar reciprocal of a circle is a conic section, of which the origin is the focus, the line corresponding to the centre is the directrix, and which is an ellipse, hyperbola, or parabola, according as the origin is within, without, or on the circle.*

299. We shall now deduce some properties concerning angles, by the help of the theorem given in Art. 297.

Any two tangents to a circle make equal angles with their chord of contact.

The line drawn from the focus to the intersection of two tangents bisects the angle subtended at the focus by their chord of contact.

For the angle between one tangent PQ (see fig. p. 257) and the chord of contact PP′ is equal to the angle subtended at the focus by the corresponding points p, q; and similarly, the angle QP′P is equal to the angle subtended by p', q; therefore, since QPP′ = QP′P, $pOq = p'Oq$.

Any tangent to a circle is perpendicular to the line joining its point of contact to the centre.

Any point on a conic, and the point where its tangent meets the directrix, subtend a right angle at the focus.

This follows as before, recollecting that the directrix of the conic answers to the centre of the circle.

Any line is perpendicular to the line joining its pole to the centre of the circle.

The line joining any point to the centre of a circle, makes equal angles with the tangents through that point.

The locus of the intersection of tangents to a circle, which cut at a given angle, is a concentric circle.

The envelope of the chord of contact of tangents which cut at a given angle is a concentric circle.

If from a fixed point tangents be drawn to a series of concentric circles, the locus of the points of contact will be a circle passing through the fixed point, and through the common centre.

Any point and the intersection of its polar with the directrix subtend a right angle at the focus.

If the point where any line meets the directrix be joined to the focus, the joining line will bisect the angle between the focal radii to the point where the given line meets the curve.

The envelope of a chord of a conic, which subtends a given angle at the focus, is a conic having the same focus and the same directrix.

The locus of the intersection of tangents, whose chord subtends a given angle at the focus, is a conic having the same focus and directrix.

If a fixed line intersect a series of conics having the same focus and same directrix, the envelope of the tangents to the conics, at the point where this line meets them, will be a conic having the same focus, and touching both the fixed line and the common directrix.

In the latter theorem, if the fixed line be at infinity, we find the envelope of the asymptotes of a series of hyperbolæ having the same focus and same directrix, to be a parabola having the same focus and touching the common directrix.

If two chords at right angles to each other be drawn through any point on a circle, the line joining their extremities passes through the centre of the circle.

The locus of the intersection of tangents to a parabola which cut at right angles is the directrix.

We say a parabola, for, the point through which the chords of the circle are drawn being taken for origin, the polar of the circle is a parabola (Art. 298).

The envelope of a chord of a circle which subtends a given angle at a given point on the curve is a concentric circle.

Given base and vertical angle of a triangle, the locus of vertex is a circle

The locus of the intersection of tangents to a parabola which cut at a given angle, is a conic having the same focus and the same directrix.

Given in position two sides of a triangle, and the angle subtended by the

passing through the extremities of the base.

The locus of the intersection of tangents to an ellipse or hyperbola which cut at right angles is a circle.

base at a given point, the envelope of the base is a conic, of which that point is a focus, and to which the two given sides will be tangents.

The envelope of any chord of a conic which subtends a right angle at any fixed point is a conic, of which that point is a focus.

" If from any point on the circumference of a circle perpendiculars be let fall on the sides of any inscribed triangle, their three feet will lie in one right line" (Art. 103).

If we take the fixed point for origin, to the triangle *inscribed* in a *circle* will correspond a triangle *circumscribed* about a *parabola;* again, to the foot of the perpendicular on any line corresponds a line through the corresponding point perpendicular to the radius vector from the origin. Hence, " If we join the focus to each vertex of a triangle circumscribed about a parabola, and erect perpendiculars at the vertices to the joining lines, those perpendiculars will pass through the same point." If, therefore, a circle be described, having for diameter the radius vector from the focus to this point, it will pass through the vertices of the circumscribed triangle. Hence, *given a triangle circumscribing a parabola, the locus of the focus is the circumscribing circle* (Art. 224, Cor. 4).

The locus of the foot of the perpendicular (or of a line making a constant angle with the tangent) from the focus of an ellipse or hyperbola on the tangent is a circle.

If from any fixed point a radius vector be drawn to a circle, and a perpendicular to it at its extremity (or a line making a constant angle with it), the envelope of this line is a conic having the fixed point for its focus.

300. Having sufficiently exemplified in the last Article the method of transforming theorems involving angles, we proceed to show that theorems involving the magnitude of lines *passing through the origin* are easily transformed by the help of the first theorem in Art. 297. For example, the sum* of the perpendiculars let fall from the origin on any pair of parallel tangents to a circle is constant, and equal to the diameter of the circle.

* Or the difference, if the origin be without the circle.

"Now, to two parallel lines correspond two points on a line passing through the origin." Hence, "the sum of the reciprocals of the segments of any focal chord of an ellipse is constant."

We know (Art. 194) that this sum is the reciprocal of the semiparameter of the ellipse, and since we learn from the present example that it only depends on the diameter, and not on the position of the reciprocal circle, we infer that *the reciprocals of equal circles, with regard to any origin, have the same parameter.*

The rectangle under the segments of any chord of a circle through the origin is constant.

The rectangle under theperpendiculars let fall from the focus on two parallel tangents is constant.

Hence, given the tangent from the origin to a circle, we are given the conjugate axis of the reciprocal hyperbola.

Again, the well-known theorem, that the sum of the focal distances of any point on an ellipse is constant, may be expressed thus:

The sum of the distances from the focus of the points of contact of parallel tangents is constant.

The sum of the reciprocals of perpendiculars let fall from any point on two tangents to a circle, whose chord of contact passes through the point, is constant.

301. Many relations involving the magnitude of lines *not* passing through the origin may be transformed by the help of the theorem (p. 95) used in Art. 298. Thus we know, that if PA, PB, PC, PD, be the perpendiculars let fall from any point of a conic on the sides of an inscribed quadrilateral, $PA.PC = kPB.PD$ (Art. 261); now we may write this relation, $\frac{PA}{OP}\cdot\frac{PC}{OP} = k.\frac{PB}{OP}\cdot\frac{PD}{OP}$, but if a, b, c, d, be the points corresponding to the lines A, B, C, D, and ap the perpendicular let fall from a on the line corresponding to P we have (p. 95) $\frac{PA}{OP} = \frac{ap}{Oa}$. Similarly for the other sides; and Oa, Ob, Oc, Od, being constant, we infer that *if a fixed quadrilateral be circumscribed to a conic, the product of the perpendiculars let fall from two opposite vertices on any variable tangent is in a constant ratio to the product of the perpendiculars let fall from the other two vertices.*

The product of the perpendiculars from any point of a conic on two fixed tangents, is in a constant ratio to the square of the perpendicular on their chord of contact. (Art. 262.)

The product of the perpendiculars from two fixed points of a conic on any tangent, is in a constant ratio to the square of the perpendicular on it, from the intersection of tangents at those points.

If, however, the origin be taken on the chord of contact, the reciprocal theorem will be, "the rectangle under the intercepts, made by any variable tangent on two parallel tangents is constant."

The product of the perpendiculars on any tangent of a conic from two fixed points (the foci) is constant.

The square of the radius vector from any fixed point to any point on a conic, is in a constant ratio to the product of the perpendiculars let fall from that point of the conic on two fixed right lines.

302. Very many theorems concerning magnitude may be reduced to theorems concerning lines cut harmonically or anharmonically, and are transformed by the following principle: *To any four points on a right line correspond four lines passing through a point, and the anharmonic ratio of this pencil is the same as that of the four points.*

This is evident, since each leg of the pencil drawn from the origin to the given points is perpendicular to one of the corresponding lines. We may thus derive the anharmonic properties of the conics in general from that of the circle.

The anharmonic ratio of the pencil joining four points on a conic to a variable fifth is constant.

The anharmonic ratio of the points in which four fixed tangents to a conic cut any variable fifth is constant.

The first of these theorems is true for the circle, since all the angles of the pencil are constant, therefore its reciprocal, the second, must be true for all the conics. The second theorem is true for the circle, since the angles which the four points subtend at the centre are constant, therefore the first theorem is true for all the conics. By observing the angles which correspond in the reciprocal figure to the angles which are constant in the case of the circle, the student will perceive that the angles which the four points of the variable tangent subtend at either focus are constant,

and that the angles are constant which are subtended at the focus by the four points in which any inscribed pencil meets the directrix.

In like manner, the theorem of Art. 144 is the reciprocal of that in Art. 142, and both, being true for the circle, must be true for all the conics.

303. The anharmonic ratio of a line is not the only relation concerning the magnitude of lines which can be expressed in terms of the angles subtended by the lines at a fixed point. In the case of anharmonic ratio $\frac{AB \cdot CD}{AD \cdot BC}$ we substitute for each line AB, $\frac{OA \cdot OB \cdot \sin AOB}{OP}$, where OP is the perpendicular on AB from the point O (note, p. 52); we find that the ratio reduces to a relation between the sines of angles subtended at O; and we infer that the ratio is the same for any four points which subtend the same angle at O. Now, if any other relation be such that it can be reduced by a similar substitution to a relation between the sines of angles subtended at a given point, this relation will be equally true for any transversal cutting the lines joining the given point to the extremities of the lines, and by taking the given point for origin a reciprocal theorem can be easily obtained. For example, the following theorem, due to Carnot, is an immediate consequence of Art. 145: "If any conic meet the side AB of any triangle in the points c, c'; BC in a, a'; AC in b, b'; then the ratio

$$\frac{Ac \cdot Ac' \cdot Ba \cdot Ba' \cdot Cb \cdot Cb'}{Ab \cdot Ab' \cdot Bc \cdot Bc' \cdot Ca \cdot Ca'} = 1."$$

Now, it will be seen that this ratio is such that we may substitute for each line Ac the angle AOc, which it subtends at any fixed point; and if we take the reciprocal of this theorem, we obtain the theorem given already at p. 244.

304. Having shown how to form the reciprocals of particular theorems, we shall add some general considerations respecting reciprocal conics.

We proved (Art. 298) that the reciprocal of a circle is an ellipse, hyperbola, or parabola, according as the origin is within,

without, or on the curve; we shall now extend this conclusion to all the conic sections. It is evident that, the nearer any line or point is to the origin, the farther the *corresponding* point or line will be; that if any line passes through the origin the corresponding point must be at an infinite distance; and that the line corresponding to the origin itself must be altogether at an infinite distance. To two tangents, therefore, through the origin on one figure, will correspond two points at an infinite distance on the other; hence, if two *real* tangents can be drawn from the origin, the reciprocal curve will have two *real* points at infinity, that is, it will be a hyperbola; if the tangents drawn from the origin be imaginary, the reciprocal curve will be an ellipse; if the origin be on the curve, the tangents from it coincide (Art. 130), therefore the points at infinity on the reciprocal curve coincide, that is, the reciprocal curve will be a parabola. Since the line at infinity corresponds to the origin, we see that, if the origin be a point on one curve, the line at infinity will be a tangent to the reciprocal curve; and we are again led to the theorem (Art. 251) that *every parabola has one tangent situated at an infinite distance.*

Hence the theorem, Art. 234, is the reciprocal of the theorem, Art. 222.

305. To the points of contact of two tangents through the origin must correspond the tangents at the two points at infinity on the reciprocal curve, that is to say, the asymptotes of the reciprocal curve. The eccentricity of the reciprocal hyperbola depending solely on the angle between its asymptotes, depends, therefore, on the angle between the tangents drawn from the origin to the original curve.

Again, the intersection of the asymptotes of the reciprocal curve (*i. e.* its centre) corresponds to the chord of contact of tangents from the origin to the original curve; or we might have otherwise inferred, by Art. 295, that to the origin and its polar, with regard to one curve, must correspond the line at infinity and its pole with regard to the other. We met with a particular case of this theorem when we proved that to the centre of a circle corresponds the directrix of the reciprocal conic, for the directrix is the polar of the origin which is the focus of that conic.

We can thus, likewise, find the *axes* of the reciprocal curve, for they must be lines drawn through its centre parallel to the internal and external bisectors of the angle between the tangents drawn from the origin. This may otherwise be expressed (by the help of the theorem, p. 166), that if through the origin we draw a conic confocal to the given one, the axes of the reciprocal conic will be parallel to the tangent and normal at the origin to the confocal conic. This latter statement is preferable, because it holds when the origin is within the curve.

306. Hence, given two circles, we can find a point such that the reciprocals of both shall be *confocal* conics. For, since the reciprocals of all circles must have one focus (the origin) common, in order that the other focus should be common, it is only necessary that the two reciprocal curves should have the same centre, that is, that the polar of the origin with regard to both circles should be the same, or that the origin should be one of the two points determined in Art. 113. Hence, given a *system* of circles, as in Art. 111, their reciprocals with regard to one of these limiting points will be a *system* of confocal conics. Theorems, therefore, concerning confocal conics, are at once transformed into theorems relating to the system of circles, *e. g.*, the theorem of Art. 187 corresponds to "the common tangent to two circles subtends a right angle at either of the limiting points." The theorem of Art. 189 corresponds to "if any line intersect two circles, its two intercepts between the circles subtend equal angles at either limiting point." Or, again, by Art. 237, any fixed point, and the fixed point through which (Art. 112) its polar must pass, subtend a right angle at the limiting points.

We may mention here that the method of reciprocal polars affords a simple solution of the problem, "to describe a circle touching three given circles." The locus of the centre of a circle touching *two* of the given circles (1), (2), is evidently a hyperbola, of which the centres of the given circles are the foci, since the problem is at once reduced to "Given base and difference of sides of a triangle." Hence (Art. 298) the polar of the centre with regard to either of the given circles (1) will always touch a circle which can be easily constructed. In like manner, the polar

of the centre of any circle touching (1) and (3) must also touch a given circle. Therefore, if we draw a common tangent to the two circles thus determined, and take the pole of this line with respect to (1), we have the centre of the circle touching the three given circles.*

307. Given any two conics, there are three points such that their reciprocals with regard to any of them will be concentric curves. For *there are three points whose polars with regard to the two conics are the same*, namely, if we form the common inscribed quadrilateral by joining the four points in which the curves intersect, the three points E, F, O (see Art. 236). These three points may be real, even when the conics cut in imaginary points.

308. *To find the equation of the reciprocal of a conic with regard to its centre.*

We found, in Art. 173, that the perpendicular on the tangent could be expressed in terms of the angles it makes with the axes

$$p^2 = a^2\cos^2\theta + b^2\sin^2\theta.$$

Hence the polar equation of the reciprocal curve is

$$\frac{k^4}{\rho^2} = a^2\cos^2\theta + b^2\sin^2\theta,$$

or

$$\frac{a^2x^2}{k^4} + \frac{b^2y^2}{k^4} = 1,$$

a concentric conic, whose axes are the reciprocals of the given one.

309. *To find the equation of the reciprocal of a conic with regard to any point* $(x'y')$.

The length of the perpendicular from any point is (Art. 173)

$$p = \frac{k^2}{\rho} = x'\cos\theta + y'\sin\theta - \sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)};$$

therefore, the equation of the reciprocal curve is

$$(xx' + yy' - k^2)^2 = a^2x^2 + b^2y^2.$$

If it were required to find the reciprocal of a conic given by its

* This solution is taken from *Gergonne's Annales.*

most general equation, this is only to find the envelope of the right line $xx' + yy' - k^2$, x' and y' being connected by the relation

$$Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F = 0.$$

Multiplying the first three terms of this last equation by k^4, the next two by $k^2(xx' + yy')$, and the last by $(xx' + yy')^2$, we find

$$(Ak^4 + Dk^2x + Fx^2)x'^2 + (Bk^4 + Dk^2y + Ek^2x + 2Fxy)x'y' + (Ck^4 + Ek^2y + Fy^2)y'^2 = 0.$$

The required envelope is, therefore (Art. 280),

$$(Bk^4+Dk^2y+Ek^2x+2Fxy)^2=4(Ak^4+Dk^2x+Fx^2)(Ck^4+Ek^2y+Fy^2);$$

or, arranging the terms,

$$(E^2 - 4CF)x^2 - 2(DE - 2BF)xy + (D^2 - 4AF)y^2 + 2(BE - 2CD)k^2x + 2(BD - 2AE)k^2y + (B^2 - 4AC)k^4 = 0.$$

It is easy to deduce from this equation the properties which we have already obtained geometrically, such as, that if the curve be a parabola, the origin will be a point on the reciprocal curve, &c.

310. Given the reciprocal of a curve with regard to the origin of co-ordinates, to find the equation of its reciprocal with regard to any point $(x'y')$.

If the perpendicular from the origin on the tangent be P, the perpendicular from any other point is (Art. 31)

$$x' \cos\theta + y' \sin\theta - P,$$

and, therefore, the polar equation of the locus is

$$\frac{k^2}{\rho} = x' \cos\theta + y' \sin\theta - \frac{k^2}{R};$$

hence

$$\frac{k^2}{R} = \frac{x'x + y'y - k^2}{\rho}, \text{ and } \frac{R\cos\theta}{k^2} = \frac{\rho\cos\theta}{xx' + yy' - k^2};$$

we must, therefore, substitute, in the equation of the curve for x, $\frac{k^2x}{xx' + yy' - k^2}$, and for y, $\frac{k^2y}{xx' + yy' - k^2}$.

The effect of this substitution may be very simply written as follows: Let the equation of the reciprocal with regard to the origin be $u_n + u_{n-1} + u_{n-2}$, &c. (see Art. 270),

then the reciprocal with regard to any point is

$$u_n + u_{n-1}\left(\frac{xx' + yy' - k^2}{k^2}\right) + u_{n-2}\left(\frac{xx' + yy' - k^2}{k^2}\right)^2 + \&c.$$

a curve of the same degree as the given reciprocal.

311. Before quitting the subject of reciprocal polars we wish to mention a class of theorems, for the transformation of which M. Chasles has proposed to take as the auxiliary conic a *parabola* instead of a *circle*. We proved (Art. 213) that the intercept made on the axis of the parabola between any two lines is equal to the intercept between perpendiculars let fall on the axis from the poles of these lines. This principle, then, enables us readily to transform theorems which relate to the magnitude of lines measured parallel to a fixed line. We shall give one or two specimens of the use of this method, premising that to two tangents parallel to the axis of the auxiliary parabola correspond the two points at infinity on the reciprocal curve, and that, consequently, the curve will be a hyperbola or ellipse, according as these tangents are real or imaginary. The reciprocal will be a parabola if the axis pass through a point at infinity on the original curve.

"Any variable tangent to a conic intercepts portions on two parallel tangents whose rectangle is constant."

To the two points of contact of parallel tangents answer the asymptotes of the reciprocal hyperbola, and to the intersection of those parallel tangents with any other tangent answer parallels to the asymptotes through any point; and we obtain, in the first instance, that the asymptotes and parallels to them through any point on the curve intercept portions on any fixed line whose rectangle is constant. But this is plainly equivalent to the theorem: "The rectangle under parallels drawn to the asymptotes from any point on the curve is constant."

Chords drawn from two fixed points of a hyperbola to a variable third point, intercept a constant length on the asymptote.

If any tangent to a parabola meet two fixed tangents, perpendiculars from its extremities on the tangent at the vertex will intercept a constant length on that line.

This method of parabolic polars is plainly much more limited in its application than the method of circular polars, whose re-

sources in transforming theorems of magnitude M. Chasles has possibly underrated.*

HARMONIC AND ANHARMONIC PROPERTIES OF CONICS.†

312. The harmonic and anharmonic properties of conic sections admit of so many applications in the theory of these curves, that we think it not unprofitable to spend a little time in pointing out to the student the number of particular theorems either directly included in the general enunciations of these properties, or which may be inferred from them without much difficulty.

The cases which we shall most frequently consider are, when one of the four points of the right line, whose anharmonic ratio we are examining, is at an infinite distance. The anharmonic ratio of four points, A, B, C, D, being in general $= \frac{AB.CD}{AD.BC}$, if D be at an infinite distance, the ratio $\frac{CD}{AD}$ is ultimately = 1, and the anharmonic ratio becomes simply $\frac{AB}{BC}$. If the line be cut harmonically, its anharmonic ratio = 1, and if D be at an infinite distance AC is bisected. The reader is supposed to be acquainted with the geometric investigation of these and the other fundamental theorems connected with anharmonic section.

313. We shall commence with the simple theorem: "If any line drawn through a point O meet a conic in the points R′, R″, and the polar of O in R, the line OR′RR″ is cut harmonically" (Art. 142).

First. Let R″ be at an infinite distance; then the line OR must be bisected at R′; that is, *if through a fixed point a line be*

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* We shall in the next part return to consider the principle of duality involved in the theory of reciprocal polars, from a purely analytical point of view; and shall show that instead of, as in this method, deriving one of two reciprocal theorems from the other, both may be regarded as equally entitled to be considered the geometrical interpretation of the same equation.

† The discovery of the anharmonic properties of conics is due to M. Chasles, from the notes to whose History of Geometry the following pages have been developed. We concur in M. Chasles's opinion, that these properties form the centre whence the whole theory of conic sections can most naturally be deduced.

drawn parallel to an asymptote of an hyperbola, or to a diameter of a parabola, the portion of this line between the fixed point and its polar will be bisected by the curve (Art. 213).

Secondly. Let R be at an infinite distance, and R′R″ must be bisected at O; that is, *if through any point a chord be drawn parallel to the polar of that point, it will be bisected at the point* (Art. 133).

If the polar of the point be at an infinite distance, every chord through the point meets the polar at an infinite distance, and therefore every chord is bisected. Hence this point is the centre, or *the centre may be considered as a point whose polar is at an infinite distance* (Art. 152).

Thirdly. Let the fixed point itself be at an infinite distance, then all the lines through it will be parallel, and will be bisected on the polar of the fixed point. Hence *every diameter of a conic may be considered as the polar of the point at infinity in which its ordinates are supposed to intersect* (Art. 285).

This also follows from the equation of the polar of a point (Art. 139),

$$(2Ax + By + D) + (2Cy + Bx + E)\frac{y'}{x'} + \frac{Dx + Ey + 2F}{x'} = 0.$$

Now, if $x'y'$ be a point at infinity on the line $my = nx$, we must make $\frac{y'}{x'} = \frac{n}{m}$, and x' infinite, and the equation of the polar becomes

$$m(2Ax + By + D) + n(2Cy + Bx + E) = 0,$$

a diameter conjugate to $my = nx$ (Art. 135).

314. We may, in like manner, make particular deductions from the theorem (Art. 144), that the two tangents through any point, any other line through the point, and the line to the pole of this last line, form an harmonic pencil.

Thus, if one of the lines through the point be a diameter, the other will be parallel to its conjugate, and since the polar of any point on a diameter is also parallel to its conjugate, we learn that the portion between the tangents of any line drawn parallel to the polar of the point is bisected by the diameter through the point.

Again, let the point be the centre, the two tangents will be

the asymptotes. Hence *the asymptotes, together with any pair of conjugate diameters, form an harmonic pencil*, and the portion of any tangent intercepted between the asymptotes is bisected by the curve (Art. 199).

315. The anharmonic property of the points of a conic (Art. 261) gives rise to a much greater variety of particular theorems. For, the four points on the curve may be any whatever, and either one or two of them may be at an infinite distance; the fifth point O, to which the pencil is drawn, may be also either at an infinite distance, or may coincide with one of the four points, in which latter case one of the legs of the pencil will be the tangent at that point; then, again, we may measure the anharmonic ratio of the pencil by the segments on *any* line drawn across it, which we may, if we please, draw parallel to one of the legs of the pencil, so as to reduce the anharmonic ratio to a simple ratio.

The following examples being intended as a practical exercise to the student in developing the consequences of this theorem, we shall, in most cases, merely state the points whence the pencil is drawn, the line on which the ratio is measured, and the resulting theorem, recommending to the reader a closer examination of the manner in which each particular theorem is inferred from the general principle.

We use the abbreviation {O.ABCD} to denote the anharmonic ratio of the pencil OA, OB, OC, OD.

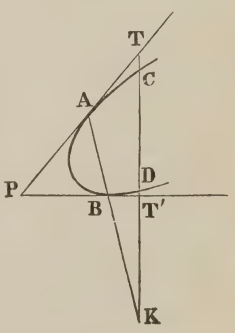
Ex. 1. {A.ABCD} = {B.ABCD}.

Let these ratios be estimated by the segments on the line CD; let the tangents at A, B, meet CD in the points T, T′, and let the chord AB meet CD in K, then the ratios are

$$\frac{TK.CD}{TD.KC} = \frac{KT'.CD}{KD.T'C},$$

that is, if any chord CD meet two tangents in T, T′, and their chord of contact in K,

$$KC.KT'.DT = KD.KT.CT'.$$



(The reader must be careful, in this and the following examples, to take the points of the pencil *in the same order* on both

sides of the equation. Thus, on the left-hand side of this equation we took K second, because it answers to the leg OB of the pencil; on the right-hand we take K first, because it answers to the leg OA).

Ex. 2. Let T and T′ coincide, then

$$KC.DT = KD.CT,$$

or, *any chord through the intersection of two tangents is cut harmonically by the chord of contact* (Art. 142).

Ex. 3. Let T′ be at an infinite distance, or the secant CD drawn parallel to PT′, and it will be found that the ratio will reduce to

$$TK^2 = TC.TD.$$

Ex. 4. Let one of the points be at an infinite distance, then $\{O.ABC\infty\}$ is constant. Let this ratio be estimated on the line $C\infty$. Let the lines AO, BO, cut $C\infty$ in a, b; then the ratio of the pencil will reduce to $\frac{Ca}{Cb}$; and we learn, that *if two fixed points*, A, B, *on a hyperbola or parabola, be joined to any variable point* O, *and the joining line meet a fixed parallel to an asymptote* (*if the curve be a hyperbola*), *or to a diameter* (*if the curve be a parabola*), *in* a, b, *then the ratio* $\frac{Ca}{Cb}$ *will be constant.*

Ex. 5. If the same ratio be estimated on any other parallel line, lines inflected from any three fixed points to a variable point cut a fixed parallel to an asymptote or diameter, so that $\frac{ab}{ac}$ is constant.

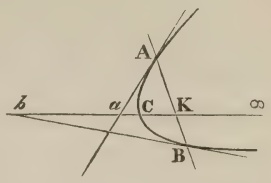
Ex. 6. It follows from Ex. 4, that if the lines joining AB to any fourth point O′ meet $C\infty$ in $a'b'$, we must have

$$\frac{ab}{a'b'} = \frac{aC}{a'C}.$$

Now let us suppose the point C to be also at an infinite distance, the line $C\infty$ becomes an asymptote, the ratio $\frac{ab}{a'b'}$ becomes one of equality, and lines joining two fixed points to any variable point on the hyperbola intercept on either asymptote a constant portion (Art. 232).

Ex. 7. $\{A.ABC\infty\} = \{B.ABC\infty\}$.

Let these ratios be estimated on $C\infty$; then if the tangents at A, B, cut $C\infty$ in a, b, and the chord of contact AB in K, we have



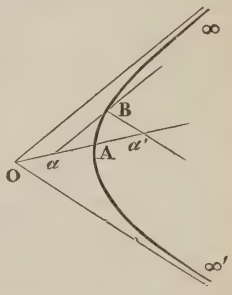
$$\frac{Ca}{CK} = \frac{CK}{Cb}.$$

(observing the caution in Ex. 1). Or, *if any parallel to an asymptote of an hyperbola, or diameter of a parabola, cut two tangents and their chord of contact, the intercept from the curve to the chord is a geometric mean between the intercepts from the curve to the tangents.* Or, conversely, if a line ab, parallel to a given one, meet the sides of a triangle in the points $a\,b\,K$, and there be taken on it a point C such that $CK^2 = Ca.Cb$, the locus of C will be a parabola, if Cb be parallel to the bisector of the base of the triangle (Art. 213), but otherwise an hyperbola, to an asymptote of which ab is parallel.

Ex. 8. Let two of the fixed points be at infinity,

$$\{\infty.AB\,\infty\,\infty'\} = \{\infty'.AB\,\infty\,\infty'\};$$

the lines $\infty\,\infty$, $\infty'\,\infty'$, are the two asymptotes, while $\infty\,\infty'$ is altogether at infinity. Let these ratios be estimated on the diameter OA; let this line meet the parallels to the asymptotes $B\,\infty$, $B\,\infty'$, in a and a'; then the ratios become $\frac{OA}{Oa} = \frac{Oa'}{OA}$. Or, *parallels to the asymptotes through any point on a hyperbola cut any semidiameter, so that it is a mean proportional between the segments on it from the centre.*



Hence, conversely, if through a fixed point O a line be drawn cutting two fixed lines, Ba, Ba', and a point A taken on it so that OA is a mean between Oa, Oa', the locus of A is a hyperbola, of which O is the centre, and Ba, Ba', parallel to the asymptotes.

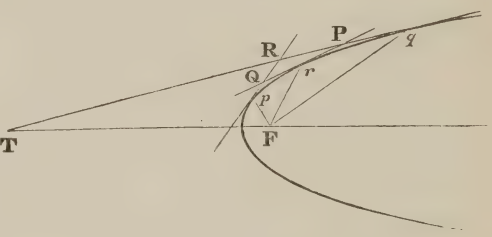
Ex. 9. $\{\infty.AB\,\infty\,\infty'\} = \{\infty'.AB\,\infty\,\infty'\}$.

Let the segments be measured on the asymptotes, and we have

$\frac{Oa}{Ob} = \frac{Ob'}{Oa'}$ (O being the centre), or *the rectangle under parallels to the asymptotes through any point on the curve is constant* (we invert the second ratio for the reason given in Ex. 1).

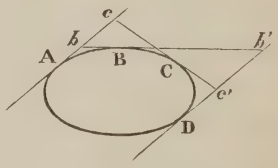
316. We next proceed to examine some particular cases of the anharmonic property of the tangents to a conic section (Art. 272).

Ex. 1. This property assumes a very simple form, if the curve be a parabola, for one tangent to a parabola is always at an infinite distance (Art. 251). Hence three fixed tangents to a parabola cut any fourth in the points A, B, C, so that $\frac{AB}{AC}$ is always constant. If the variable tangents coincide in turn with each of the given tangents, we obtain the theorem,



$$\frac{pQ}{QR} = \frac{RP}{Pq} = \frac{Qr}{rP}.$$

Ex. 2. Let two of the four tangents to an ellipse or hyperbola be parallel to each other, and let the variable tangent coincide alternately with each of the parallel tangents. In the first case the ratio is $\frac{Ab}{Ac}$, and in the second $\frac{Dc'}{Db'}$.



Hence the rectangle $Ab \,.\, Db'$ is constant.

It may be deduced from the anharmonic property of the points of a conic, that if the lines joining any point on the curve O to A, D, meet the tangents in the points b, b', then the rectangle $Ab \,.\, Db'$ will be constant.

INVOLUTION.

317. If there be three pairs of points on a right line, AA′, BB′, CC′ (of which A is said to be *conjugate* to A′, &c.), and if

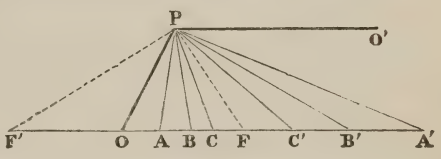
the anharmonic ratio of any four of them is equal to that of their four conjugates,

$$\{ABCA'\} = \{A'B'C'A\},$$

then the anharmonic ratio of any other four will be equal to that of *their* four conjugates, and the six points are said to be *in involution.*

M. Chasles has proved this theorem algebraically, but the following geometrical demonstration, communicated to me by Mr. Townsend, is much more simple, and appears to me to give a much clearer idea of the relation between points in involution.

It is, in general,* possible to find some point P at which AB and A′B′ will subtend equal angles, since it is only necessary to describe on AB, A′B′ segments of circles intersecting each other which contain the same angle. Now since we are given



$$\{P.ABCA'\} = \{P.A'B'C'A\},$$

and since two angles of these pencils are equal,

$$(APB = A'PB',\ \ BPA' = B'PA)$$

the remaining angle will be equal ($APC = A'PC'$); for we have

$$\frac{\sin APB \,.\, \sin CPA'}{\sin APC \,.\, \sin BPA'} = \frac{\sin A'PB' \sin C'PA}{\sin A'PC' \sin B'PA}.$$

But since $APB = A'PB'$ and $BPA' = B'PA$, we infer from this equation that PC and PC′ divide the angle APA′ into parts whose sines are in the same ratio, therefore $APC = A'PC'$, and therefore the angle any two points subtend at P is equal to that subtended by their two conjugates.

We recommend the reader to make a table of the different relations of magnitude between the six points inferred from this identity of anharmonic ratios; for instance, from

$$\{ABCA'\} = \{A'B'C'A\}$$

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* The locus of such points P is in general a circle. This locus, however, may become imaginary, as, for instance, if the points A′ B′ lie between A and B. The reader who has caught the force of the reasoning in Art. 243, will perceive that even in this case the theorem in the text does not cease to be true.

we have $$\frac{AB.CA'}{AA'.BC} = \frac{A'B'.C'A}{AA'.B'C'},$$

or $$AB.CA'.B'C' = A'B'.C'A.BC.$$

We think it unnecessary to enlarge on the development of these relations, as it can present no difficulty to the reader.

318. This method of considering points in involution shows at once that their number is not limited to six, for if we take any other point D on the line, and draw a line so that A'PD' = APD, we obtain another pair of points which with any two of the three original pairs will form six points in involution; and in like manner we can form a whole system of points in involution, any three pairs of which have the relation described above.

We see also that two pairs of points *determine* the system; that if, for example, CC', DD', EE' be each in involution with AA', BB', they will be in involution with each other; and that, given the points AA', BB', we can find the point conjugate to any other (C). For the equation just written gives the ratio B'C' : C'A in terms of known quantities, and therefore enables us to find C'. The following geometrical construction for the same purpose is only another expression of a theorem which we shall just prove, (Art. 320): "Assume any point at random d; join dC, dD', dB'; construct any triangle whose vertices rest on these three lines, and two of whose sides pass through B, D, then the remaining side will pass through C', the point conjugate to C. The point d might have been taken at infinity, and the three lines dC, dD', dB' would then be parallel.

One remarkable case deserves examination: it is when one of the points has its conjugate at an infinite distance. This will happen if we draw, PO' parallel to AA', and take a point O such that A'PO = APO'. The point O will then have its conjugate point at infinity. O is called the *centre* of the system of points in involution.

Now the relation between the points takes in this case a very simple form, for we have

$$\{ABOO'\} = \{A'B'O'O\},$$

or $$\frac{AO.BO'}{AO'.BO} = \frac{A'O'.B'O}{A'O.B'O'};$$

let O′ be at an infinite distance, and this equation becomes

$$OA.OA' = OB.OB'.$$

Or, *the product of the distances from the centre of any two conjugate points is constant.*

"Given two pairs of points of the system, to find the centre," is only a particular case of the theorem just given, and the following is the form which the construction just given will in this case assume: "Through A, B draw any pair of parallels LA, MB; through A′, B′, a different pair of parallels, MA′, LB′; then LM will pass through the centre of the system."

319. Another important particular case is when a point coincides with its conjugate. We shall term such a point a *focus** of the system of points in involution. Every system of points in involution has evidently *two foci*, namely, the points FF′, where the external and internal bisectors of the angle APA′ meet the line AA′. Hence evidently the line F′AFA′ is cut harmonically; or *any point, its conjugate, together with the two foci, form four points of a line cut harmonically.*

Given two pairs of points of the system, we can find the foci, either by the construction Art. 317, or as follows: Since F is conjugate to itself we have

$$\{AFBA'\} = \{A'FB'A\},$$

or

$$\frac{AF.BA'}{A'F.BA} = \frac{A'F.B'A}{AF.B'A'}.$$

Hence

$$AF^2 : A'F^2 :: AB.AB' : A'B.A'B';$$

or F is the point where the line AA′ is cut, either internally or externally, in a certain given ratio. It is easy to see that this ratio (and therefore the foci) will be imaginary at the same time that the locus of P is imaginary in the geometrical construction of Art. 317.

This may be otherwise solved by first finding the centre by the last Article; then if any two conjugate points AA′ be on the same side of the centre, we get the foci by taking $OA.OA' = OF^2$,

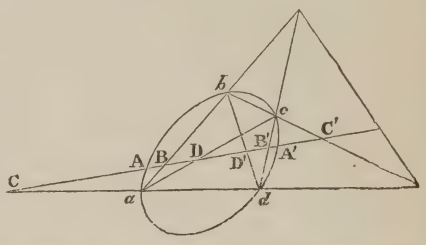
* I borrow this name from a paper published by Mr. Davies in *The Mathematician* (vol. i. pp. 169, 243), which contains the most complete account of anharmonic section and involution which I have met with in English.

but if they be on opposite sides of the centre the foci are imaginary, for no point then can coincide with its conjugate; and the conjugate of the point F such that $OA.OA' = OF^2$ will not be itself, but the point F′ equidistant from the centre.

It is important to observe that the relation between six points in involution is of the class noticed in Art. 303, and is such that the same relations will subsist between the sines of the angles subtended by them at any point as subsist between the segments of the lines themselves. Consequently, *if a pencil be drawn from any point to six points in involution, any transversal cuts this pencil in six points in involution.* Again, *the reciprocal of six points in involution is a pencil in involution.**

320. We proceed to mention the most important application of these principles to the theory of conic sections.

If a quadrilateral *abcd* be inscribed in a conic section, and any transversal cut the conic in A, A′, the sides *ab*, *cd*, in B, B′, and the sides *ad*, *bc* in C, C′, then the points AA′BB′CC′ are in involution, for by the anharmonic property of conic sections,



$$\{a.\mathrm{A}d\,b\mathrm{A}'\} = \{c.\mathrm{A}d\,b\mathrm{A}'\};$$

* Involution is itself a particular case of the following:—If we have a system of points on a right line, ABCDE, &c., and another system, A′B′C′D′E′, &c., either on the same or on a different right line, the systems are said to be *similar*, if the anharmonic ratio of any four points whatever of the first system be equal to that of the four corresponding points of the second system. Thus if we draw a pencil from any point O, to the points of the first system, and cut it by any transversal, we shall have a similar system. We can construct a system similar to a given one, and such that *three* arbitrary points A′B′C′ shall correspond to three given points ABC of the first system. For, draw through A any line making an angle with AB, on it take $Ab = A'B'$, $Ac = A'C'$, and the intersection of bB, cC, determines the point O, and, therefore, the entire of the second system.

When the two systems form part of the same right line, it will not generally happen that a given point will have the same conjugate when it is considered as belonging to the first and as belonging to the second system. If, however, two similar systems form part of the same right line, and if every point have the same conjugate, to whichever system it be considered to belong, the entire series forms a system in involution.

but if we observe the points in which these pencils cut AA′, we get

$$\{ACBA'\} = \{AB'C'A'\} = \{A'C'B'A\}.$$

If any other conics through the points *abcd* meet the transversal in points DD′, EE′, &c., since each pair of points is in involution with BB′, CC′, these points form a system in involution. Hence *a system of conics circumscribing the same quadrilateral meet any transversal in a system of points in involution.*

Reciprocally, *if a system of conics be inscribed in the same quadrilateral, the pairs of tangents drawn to them from any point will form a system in involution.*

321. The following particular inferences are drawn from this property by the help of Art. 319:

If three conics circumscribe the same quadrilateral, the common tangent to any two will be cut harmonically by the third. For the points of contact of these tangents will be the foci of the system in involution.

Again, if one of the conics break up into two right lines, we learn that, *if through the intersection of the common chords of two conics we draw a tangent to one of them, this line will be cut harmonically by the other.*

Or again, if we suppose the two common chords to coincide, we learn that *if two conics have double contact with each other, or if they have a contact of the third order, any tangent to the one is cut harmonically at the points where it meets the other, and where it meets the chord of contact.*

Thus likewise we can describe a conic through four points *abcd* to touch a given right line; for the point of contact must be one of the foci of the system BB′CC′, &c., and these points can be determined by Art. 319. This problem, therefore, admits of two solutions.

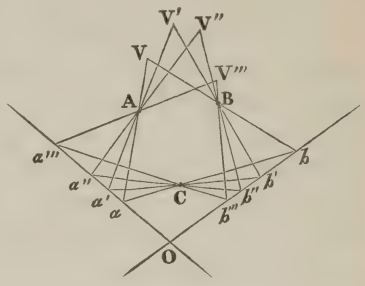
322. We now proceed to give some examples of problems easily solved by the help of the anharmonic properties of conic sections.

Ex. 1. To prove Mac Laurin's method of generating conic sections (p. 235), viz., *To find the locus of the vertex* V *of a triangle*

whose sides pass through the points A, B, C, *and whose base angles move on the fixed lines* O*a*, O*b*.

Let us suppose four such triangles drawn, then since the pencil $\{C.aa'a''a'''\}$ is the same pencil as $\{C.bb'b''b'''\}$, we have

$$\{aa'a''a'''\} = \{bb'b''b'''\},$$



and, therefore,

$$\{A.aa'a''a'''\} = \{B.bb'b''b'''\};$$

or, from the nature of the question,

$$\{A.VV'V''V'''\} = \{B.VV'V''V'''\};$$

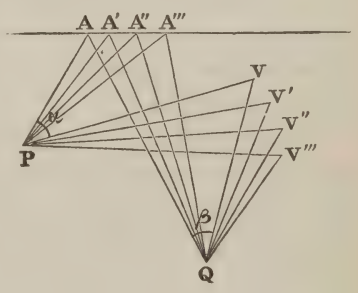
and therefore A, B, V, V′, V″, V‴ lie on the same conic section. Now if the first three triangles be fixed, it is evident that the locus of V‴ is the conic section passing through ABVV′V″.

Ex. 2. M. Chasles has showed that the same demonstration will hold if the side *ab*, instead of passing through the fixed point C, touch any conic which touches O*a*, O*b*, for then any four positions of the base cut O*a*, O*b*, so that

$$\{aa'a''a'''\} = \{bb'b''b'''\} \text{ (Art. 272)},$$

and the rest of the proof proceeds the same as before.

Ex. 3. Newton's method of generating conic sections. *Two angles of constant magnitude move about fixed points* P, Q, *the intersection of two of their sides traverses the right line* AA′; *then the locus of* V, *the intersection of their other two sides, will be a conic passing through* P, Q.



For, as before, take four positions of the angles, then

$$\{P.AA'A''A'''\} = \{Q.AA'A''A'''\};$$

but

$$\{P.AA'A''A'''\} = \{P.VV'V''V'''\},$$

since the angles of the pencils are the same; and

$$\{Q.AA'A''A'''\} = \{Q.VV'V''V'''\},$$

therefore

$$\{P.VV'V''V'''\} = \{Q.VV'V''V'''\};$$

and, therefore, as before, the locus of V‴ is a conic through P, Q, V, V′, V″.

Ex. 4. M. Chasles has extended this method of generating conic sections, by supposing the point A, instead of moving on a right line, to move on any conic passing through the points PQ, for we shall still have

$$\{P.AA'A''A'''\} = \{Q.AA'A''A'''\}.*$$

Ex. 5. The demonstration would be the same if, in place of the angles APV, AQV being constant, APV and AQV cut off constant intercepts each on one of two fixed lines, for we should then prove the pencil

$$\{P.AA'A''A'''\} = \{P.VV'V''V'''\},$$

because both pencils cut off intercepts of the same length on a fixed line.

Thus, also, given base of a triangle and the intercept made by the sides on any fixed line, we can prove that the locus of vertex is a conic section.

Ex. 6. We may also extend Ex. 1, by supposing the extremities of the line *ab* to move on any conic section passing through

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* In consequence of some doubts expressed by Mr. Davies in his Paper, quoted p. 277, as to the validity of this demonstration, I subjoin the following proof by the method of co-ordinates. Let the middle point of the line PQ be taken for origin, and PQ for axis of x, let A be $x'y'$, V, xy, then we have

$$\frac{\frac{y'}{c+x'} - \frac{y}{c+x}}{1+\frac{yy'}{(c+x)(c+x')}} = \tan\alpha, \quad \text{and} \quad \frac{\frac{y'}{c-x'} - \frac{y}{c-x}}{1+\frac{yy'}{(c-x)(c-x')}} = \tan\beta,$$

which are equivalent to equations of the form

$$Ax' + By' + Ac = 0, \qquad A'x' + B'y' - A'c = 0,$$

where A, B, A′, B′, are linear functions of x and y; hence we have

$$x' = \frac{BA' + AB'}{BA' - AB'}c, \quad y' = \frac{2AA'}{AB' - BA'}c.$$

Now, if $x'y'$ be restricted to move on a conic section, substituting these values for x', y', we find an equation of the fourth degree; but if *this conic pass through the points* PQ, since, when we make $y' = 0$, the equation of the conic must become $x'^2 - c^2 = 0$, the equation of the generated curve will become divisible by AA′, and is only *of the second degree*. The student will not find it difficult to explain why A, A′, enter as factors into the equation of the locus, if he examine the geometric signification of those quantities.

the points AB, for, taking four positions of the triangle, we have, by Art. 272, $\{aa'a''a'''\} = \{bb'b''b'''\}$;
therefore, $\{A.aa'a''a'''\} = \{B.bb'b''b'''\}$,
and the rest of the proof proceeds as before.

Ex. 7. The base of a triangle passes through C, the intersection of common tangents to two conic sections; the extremities of the base ab lie one on each of the conic sections, while the sides pass through fixed points AB, one on each of the conics: the locus of the vertex is a conic through A, B.

The proof proceeds exactly as before, depending now on the last theorem proved Art. 273.

We may mention that the theorem of Art. 273 admits of a simple geometrical proof. Let the pencil $\{O.ABCD\}$ be drawn from points corresponding to $\{o.abcd\}$. Now, the lines OA, oa, intersect at r on one of the common chords of the conics; in like manner, BO, bo, intersect in r' on the same chord, &c.; hence $\{rr'r''r'''\}$ measures the anharmonic ratio of both these pencils.

Ex. 8. In Ex. 6 the base, instead of passing through a fixed point C, may be supposed to touch a conic having double contact with the given conic (see Art. 275).

Ex. 9. If a polygon be inscribed in a conic, all whose sides but one pass through fixed points, the envelope of that side will be a conic having double contact with the given one.

For, take any four positions of the polygon, then, if a, b, c, &c., be the vertices of the polygon, we have

$$\{aa'a''a'''\} = \{bb'b''b'''\} = \{cc'c''c'''\}, \text{ &c.}$$

The problem is, therefore, reduced to that of Art. 275, "Given three pairs of points, $aa'a''$, $dd'd''$, to find the envelope of $a'''d'''$, such that $\{aa'a''a'''\} = \{dd'd''d'''\}$."

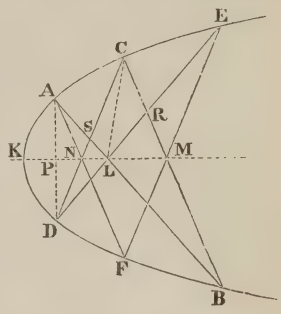
Ex. 10. To inscribe a polygon in a conic section, all whose sides pass through fixed points.

If we assume any point (a) at random on the conic for the vertex of the polygon, and form a polygon whose sides pass through the given points, the point z, where the last side meets the conic, will not, in general, coincide with a. If we make four such attempts to inscribe the polygon, we must have, as in the last example, $\{aa'a''a'''\} = \{zz'z''z'''\}$.

Now, if the last attempt were successful, the point a''' would coincide with z''', and the problem is reduced to, "Given three pairs of points, $aa'a''$, $zz'z''$, to find a point K such that

$$\{Kaa'a''\} = \{Kzz'z''\}."$$

Now, if we make $az''a'za''z'$ the vertices of an inscribed hexagon (in the order here given, taking an a and z alternately, and so that az, $a'z'$, $a''z''$, may be opposite vertices), then either of the points in which the line joining the intersections of opposite sides meets the conic may be taken for the point K. For, in the figure, the points ACE are $aa'a''$, DFB are $zz'z''$, and if we take the sides in the order ABCDEF, L, M, N, are the intersections of opposite sides. Now, since {KPNL} measures both {D.KACE} and {A.KDFB}, we have



$$\{KACE\} = \{KDFB\} \quad \text{Q. E. D.}^*$$

It is easy to see, from the last example, that K is a point of contact of a conic having double contact with the given conic, to which az, $a'z'$, $a''z''$ are tangents, and that we have therefore just given the solution of the question, "to describe a conic touching three given lines, and having double contact with a given conic."

Ex. 11. The anharmonic property affords also a simple proof of Pascal's theorem, alluded to in the last example.

We have {E.CDFB} = {A.CDFB}. Now, if we examine the segments made by the first pencil on BC, and by the second on DC, we have $\{CRMB\} = \{CDNS\}$.

Now, if we draw a pencil from the point L to each of these points, both pencils will have the three legs, CL, DE, AB, common, therefore the fourth legs, NL, LM, must form one right line (Art. 317).

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* This construction for inscribing a polygon in a conic is due to M. Poncelet (*Traité des Propriétés Projectives*, p. 351). The demonstration here used, which was communicated to me by Mr. Townsend, seems to me more simple than that employed by M. Poncelet. The proof here used shows that Poncelet's construction will equally solve the problem, "To inscribe a polygon in a conic, each of whose sides shall touch a conic having double contact with the given conic." The conics touched by the sides may be all different.

Ex. 12. Pascal's theorem leads at once to Mac Laurin's method of generating conic sections, for if we suppose the five points ABCDE given, and F variable, then F will be the vertex of a triangle FMN, whose sides pass through the fixed points L, A, E, and whose base angles move on the fixed lines CD, CB. We see, therefore, that, *given five points on a conic, we can determine as many other points on the conic as we please.* By the same construction, *given five points on a conic*, ABCDE, *we can determine the point where any line* AN *through one of them meets the conic again.* So also, *given five points on a conic we can find its centre.* For we may draw parallels through A to BC, BD, and determine the points where they meet the conic again, and then find the centre by note, p. 130.

Ex. 13. *Given four points on a conic*, ADFB, *and two fixed lines through any one of them*, DC, DE, *to find the envelope of the line* CE *joining the points where those fixed lines again meet the curve.*

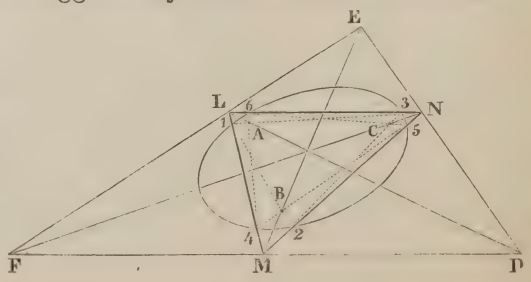
The vertices of the triangle CEM move on the fixed lines DC, DE, NL, and two of its sides pass through the fixed points, B, F, therefore, the third side envelopes a conic section touching DC, DE (by the reciprocal of Mac Laurin's mode of generation).

Ex. 14. *Given four points on a conic* ABDE, *and two fixed lines*, AF, CD, *passing each through a different one of the fixed points, the line* CF *joining the points where the fixed lines again meet the curve will pass through a fixed point.*

For the triangle CFM has two sides passing through the fixed points B, E, and the vertices move on the fixed lines AF, CD, NL, which fixed lines *meet in a point*, therefore (p. 254) CF passes through a fixed point.

The reader will find, in the section on Projections, how the last two theorems are suggested by other well-known theorems.

Ex. 15. To inscribe a triangle in a conic whose three sides pass through three given points.



This is of course a particular case of Ex. 10, but our pre-

sent object is to give a geometrical proof of the construction used at p. 234.

If we consider the quadrilateral of which E, L, N are vertices, and D, F the intersections of opposite sides; by the harmonic properties of a quadrilateral, ML, ME, MN, MD form a harmonic pencil, and therefore the line B1 is cut harmonically in the points where it meets these four lines. But since B is the pole of MD, B1 is also cut harmonically in the points where it meets the conic and where it meets MD; hence it appears that B1 and MN must intersect on the conic, or that 1, 2, B lie on one right line. In the same manner it is proved that 13 passes through A, and 32 through C.

323. *The anharmonic ratio of four points on a right line is the same as that of their polars with respect to any conic.*

For this property is true for the circle (Art. 302), and if we take the reciprocal of this circle with regard to any point, we see, by Art. 302, that the property must be true for any conic. A particular case of this theorem is, *the anharmonic ratio of any four diameters is equal to that of their four conjugates.* We might also prove this directly, from the consideration that the anharmonic ratio of four chords proceeding from any point of the curve is equal to that of the supplemental chords (Art. 174).

A conic circumscribes a given quadrilateral, to find the locus of its centre.

Draw diameters of the conic bisecting the sides of the quadrilateral, their anharmonic ratio is equal to that of their four conjugates, but this last ratio is given, since the conjugates are parallel to the four given lines; hence the locus is a conic passing through the middle points of the given sides. If we take the cases where the conics break up into two right lines, we see that the intersections of the diagonals, and also those of the opposite sides, are points in the locus, and, therefore, that these points lie on a conic passing through the middle points of the sides and of the diagonals. When the given quadrilateral has a re-entrant angle it is easy to see that such a quadrilateral cannot be inscribed in a closed figure of the shape of the ellipse or parabola, and that the circumscribing conic must therefore be a hyperbola, which

may have some of the vertices in opposite branches. But since the centre of an hyperbola is never at infinity, the locus of centres must in this case be an *ellipse*. Through four points not so disposed in general two parabolæ can be drawn, for (Art. 253) this is a particular case of the last problem of Art. 321. The locus of centres will in this case be a *hyperbola*, having for asymptotes lines parallel to the diameters of these two parabolæ. The locus of centres will be a *parabola* when one of the given points is at an infinite distance; that is, when it is required, "given three points and a parallel to an asymptote, to find the locus of centre."

It is very easy to show, by the same method, that the locus of the pole of any given right line is a conic section.

324. We think it unnecessary to go through the theorems, which are only the polar reciprocals of those investigated in the last examples, but we recommend the student to form the polar reciprocal of each of these theorems, and then to prove it directly by the help of the anharmonic property of the *tangents* of a conic. A single example will suffice.

Any transversal through a fixed point P *meets two fixed lines* OA, OB, *in the points* A, B, *and portions of given lengths* AC, BD, *are taken on those lines: to find the envelope of* CD.

Take any four positions of the transversal, and we have

$$\{AA'A''A'''\} = \{BB'B''B'''\},$$

but $\{AA'A''A'''\} = \{CC'C''C'''\}$, and $\{BB'B''B'''\} = \{DD'D''D'''\}$; therefore, the four lines, CD, C′D′, C″D″, C‴D‴, cut the two lines OC, OD, so that

$$\{CC'C''C'''\} = \{DD'D''D'''\},$$

and, therefore, the envelope of CD is a conic touching OA, OB.

325. Generally when the envelope of a moveable line is found by this method to be a conic section, it is useful to take notice whether in any particular position the moveable line can be altogether at an infinite distance, for if it can, the envelope is a parabola (Art. 253). Thus, in the last example the line CD cannot be at an infinite distance, unless in some position AB can be at an infinite distance, that is, unless P is at an infinite distance. Hence we see that in the last example if the transversal, instead

of passing through a fixed point, were parallel to a given line, the envelope would be a parabola. In like manner, the nature of the locus of a moveable point is often at once perceived by observing particular positions of the moveable point, as we have exemplified in Art. 323.

326. Given three points on a right line, a, b, c, and three points on another right line, A, B, C, if we take dD so that

$$\{abcd\} = \{\mathrm{ABCD}\},$$

it is evident, from the preceding articles, that the envelope of dD is a conic section, and that the lines pd, PD, joining dD to two fixed points, will intersect on a conic passing through these points.* Let us examine the most general relation between d and D that this should be the case. If we denote the distances of $abcd$ from any fixed point o on the same line by r, r', r'', r''', and the distances of ABCD from a fixed point O on the other right line by R, R′, R″, R‴, we have

$$\frac{(r - r')\,(r'' - r''')}{(r - r'')\,(r' - r''')} = \frac{(\mathrm{R} - \mathrm{R}')\,(\mathrm{R}'' - \mathrm{R}''')}{(\mathrm{R} - \mathrm{R}'')\,(\mathrm{R}' - \mathrm{R}''')};$$

and if we suppose r and R alone variable, this gives a relation of the form

$$k\mathrm{R}r + l\mathrm{R} + mr + n = 0 \quad \text{(compare Art. 275)}.$$

This relation containing three independent constants is, therefore, the most general connexion between od and OD if dD envelope a conic touching od, OD.

If $k = 0$, dD will envelope a parabola, since then R and r will become infinite at the same time.

M. Chasles has given this relation in a different form. Let there be given two other points e and E, then if $\lambda \cdot \frac{ed}{od} + \mu \cdot \frac{\mathrm{ED}}{\mathrm{OD}} = 1$,

* We saw, p. 232, that it is also true, if ABCD, $abcd$, be points on the same conic section, that Dd will envelope a conic if $\{\mathrm{ABCD}\} = \{abcd\}$, and the intersection of PD, pd, will in this case be a conic if P, p be points on the conic. Again, any two conics will be cut by four tangents to any conic having double contact with both, so that $\{\mathrm{ABCD}\} = \{abcd\}$ (Art. 275); but it will not be true conversely, that, if this relation holds, the envelope of Dd will be a conic, unless the points ABC, abc, be so taken that Aa, Bb, Cc, may all touch the same conic having double contact with both.

dD will envelope a conic; for if its distances eo, EO, be called a, A, this relation may be written

$$\lambda \cdot \frac{r - a}{r} + \mu \cdot \frac{R - A}{R} = 1,$$

an equation included in the general form we have given.

THE METHOD OF INFINITESIMALS.

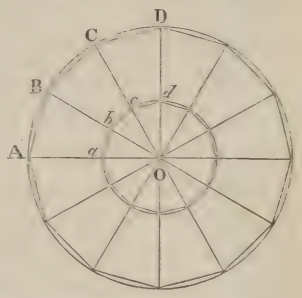
327. In the next Part we purpose to show how the differential calculus enables us readily to draw tangents to curves, and to determine the magnitude of their areas and arcs. We wish first, however, to give the reader some idea of the manner in which these problems were investigated by geometers before the invention of that method. The geometric methods are not merely interesting in a historical point of view; they afford solutions of some questions more concise and simple than those furnished by analysis, and they have even recently led to a beautiful theorem (Art. 337), which had not been anticipated by those who have applied the integral calculus to the rectification of conic sections.

If a polygon be inscribed in any curve, it is evident that the more the number of the sides of the polygon is increased, the more nearly will the area and perimeter of the polygon approach to equality with the area and perimeter of the curve, and the more nearly will any side of the polygon approach to coincidence with the tangent at the point where it meets the curve. Now, if the sides of the polygon be multiplied *ad infinitum*, the polygon will coincide with the curve, and the tangent at any point will coincide with the line joining two indefinitely near points on the curve. In like manner, we see that the more the number of the sides of a *circumscribing* polygon is increased, the more nearly will its area and perimeter approach to equality with the area and perimeter of the curve, and the more nearly will the intersection of two of its adjacent sides approach to the point of contact of either. Hence, in investigating the area or perimeter of any curve, we may substitute for the curve an inscribed or circumscribing polygon of an indefinite number of sides; we may consider any tangent of the curve as the line joining two indefinitely near points on the curve, and any point on the curve as the intersection of two indefinitely near tangents.

The nature of this method will be best understood by examples, of which we give a few, commencing with the simple case of the circle.

328. Ex. 1. *To find the direction of the tangent at any point of a circle.*

In any isosceles triangle AOB, either base angle OBA is less than a right angle by half the vertical angle; but as the points A and B approach to coincidence, the vertical angle may be supposed less than any assignable angle, therefore the angle OBA which the tangent makes with the radius, is ultimately equal to a right angle. We shall frequently have occasion to use the principle here proved, viz., that two indefinitely near lines *of equal length* are at right angles to the line joining their extremities.



Ex. 2. *The circumferences of two circles are to each other as their radii.*

If polygons of the same number of sides be inscribed in the circles, it is evident, by similar triangles, that the bases *ab*, AB, are to each other as the radii of the circles, and, therefore, that the whole perimeters of the polygons are to each other in the same ratio; and since this will be true, no matter how the number of sides of the polygon be increased, the circumferences are to each other in the same ratio.

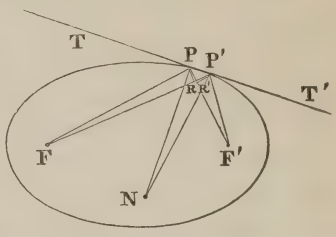
Ex. 3. *The area of a circle is equal to the radius multiplied by the semicircumference.*

For the area of any triangle OAB is equal to half its base multiplied by the perpendicular on it from the centre; hence the area of any inscribed regular polygon is equal to half the sum of its sides multiplied by the perpendicular on any side from the centre; but the more the number of sides is increased, the more nearly will the perimeter of the polygon approach to equality with that of the circle, and the more nearly will the perpendicular on any side approach to equality with the radius, and the difference between them can be made less than any assignable

quantity; hence ultimately the area of the circle is equal to the radius multiplied by the semicircumference.

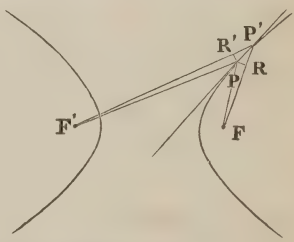
If we denote the circumference by $2\pi r$, the area will therefore $= \pi r^2$.

329. Ex. 1. *To determine the direction of the tangent at any point on an ellipse.*



Let P and P′ be two indefinitely near points on the curve, then FP + PF′ = FP′ + P′F′; or, taking FR = FP, F′R′ = F′P′, we have P′R = PR′; but in the triangles PRP′, PR′P′, we have also the base PP′ common, and (by Ex. 1, Art. 328) the angles PRP′, PR′P′ right; hence the angle PP′R = P′PR′. Now TPF is ultimately equal to PP′F, since their difference PFP′ may be supposed less than any given angle; hence TPF = P′PF′, or the focal radii make equal angles with the tangent.

Ex. 2. *To determine the direction of the tangent at any point on a hyperbola.*



We have

$$F'P' - F'P = FP' - FP,$$

or, as before,

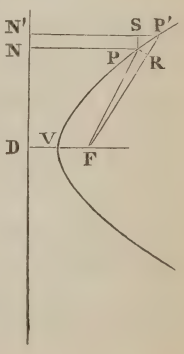
$$P'R = P'R'.$$

Hence the angle

$$PP'R = PP'R',$$

or, the tangent is the internal bisector of the angle FPF′.

Ex. 3. *To determine the direction of the tangent at any point of a parabola.*



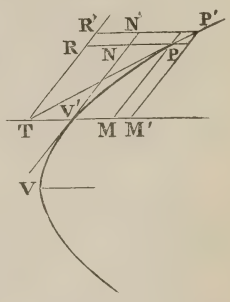
We have FP = PN, and FP′ = P′N′; hence P′R = P′S, or the angle N′P′P = FP′P. The tangent, therefore, bisects the angle FPN.

330. *To find the area of the parabolic sector* FVP.

Since PS = PR, and PN = FP, we have the triangle FPR half the parallelogram PSNN′. Now if we take a number of points P′P″, &c.

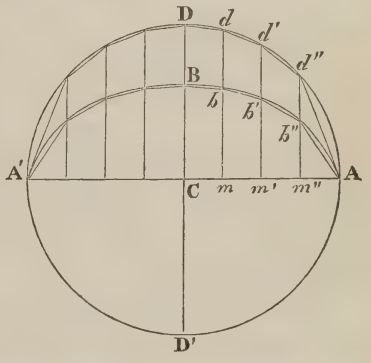
between V and P, it is evident that the closer we take them, the more nearly will the sum of all the parallelograms PSNN′, &c., approach to equality with the area DVP, and the sum of all the triangles PFR, &c., to the sector VFP; hence ultimately the sector PFV is half the area DVPN, and therefore one-third of the quadrilateral DFPN.

Ex. 2. *To find the area of the segment of a parabola cut off by any right line.*



Draw the diameter bisecting it, then the parallelogram PR′ is equal to PM′, since they are the complements of parallelograms about the diagonal; but since TM is bisected at V′, the parallelogram PN′ is half PR′; if, therefore, we take a number of points PP′P″, &c., it follows that the sum of all the parallelograms PM′ is double the sum of all the parallelograms PN′, and therefore ultimately that the space V′PM is double V′PN; hence the area of the parabolic segment V′PM is to that of the parallelogram V′NPM in the ratio 2 : 3.

331. Ex. 1. *The area of an ellipse is equal to the area of a circle whose radius is a geometric mean between the semiaxes of the ellipse.*



For if the ellipse and the circle on the transverse axis be divided by any number of lines parallel to the axis minor, then since $mb : md :: m'b' : m'd' :: b : a$, the quadrilateral $mbb'm'$ is to $mdd'm'$ in the same ratio, and the sum of all the one set of quadrilaterals, that is, the polygon $Bbb'b''A$ inscribed in the ellipse is to the corresponding polygon $Ddd'd''A$ inscribed in the circle, in the same ratio. Now this will be true whatever be the number of the sides of the polygon: if we suppose them, therefore, increased indefinitely, we learn that

the area of the ellipse is to the area of the circle as b to a; but the area of the circle being $= \pi a^2$, the area of the ellipse $= \pi ab$.

Cor. It can be proved, in like manner, that if any two figures be such that the ordinate of one is in a constant ratio to the corresponding ordinate of the other, the areas of the figures are in the same ratio.

Ex. 2. *Every diameter of a conic bisects the curve.*

For if we suppose a number of ordinates drawn to this diameter, since the diameter bisects them all, it also bisects the trapezium formed by joining the extremities of any two adjacent ordinates, and by supposing the number of these trapezia increased without limit, we see that the diameter bisects the curve.

332. Ex. 1. *The area of the sector of a hyperbola made by joining any two points of it to the centre, is equal to the area of the segment made by drawing parallels from them to the asymptotes.*

For since the triangle PKC = QLC (Art. 202) the area PQC = PQKL.

Ex. 2. *Any two segments,* PQKL, RSMN, *are equal, if*

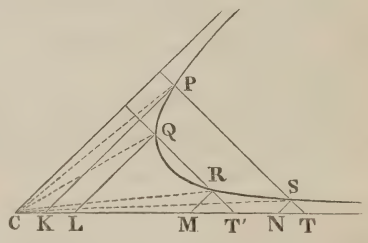
PK : QL :: RM : SN.

For

PK : QL :: CL : CK,

but (Art. 200)

CL = MT′, CK = NT;



we have, therefore,

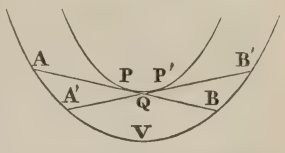
RM : SN :: MT′ : NT,

and therefore QR is parallel to PT. We can now easily prove that the sectors PCQ, RCS are equal, since the diameter bisecting PS, QR will bisect both the hyperbolic area PQRS, and also the triangles PCS, QCR.

If we suppose the points Q, R to coincide, we see that we can bisect any area PKNS by drawing an ordinate QL, a geometric mean between the ordinates at its extremities.

Again, if a number of ordinates be taken forming a continued geometric progression, the area between any two is constant.

333. *The tangent to the interior of two similar, similarly placed, and concentric conics cuts off a constant area from the exterior conic.*



For we proved (p. 203) that this tangent is always bisected at the point of contact; now if we draw any two tangents, the angle AQA′ will be equal to BQB′, and the nearer we suppose the point Q to P, the more nearly will the sides AQ, A′Q approach to equality with the sides BQ, B′Q; if, therefore, the two tangents be taken indefinitely near, the triangle AQA′ will be equal to BQB′, and the space AVB will be equal to A′VB′; since, therefore, this space remains constant as we pass from any tangent to the consecutive tangent, it will be constant whatever tangent we draw.

Cor. 1. It can be proved, in like manner, that if a tangent to one curve always cut off a constant area from another, it will be bisected at the point of contact; and, conversely, that if it be always bisected it cuts off a constant area.

Hence we can draw through a given point a line to cut off from a given conic the *minimum* area. If it were required to cut off a *given* area it would be only necessary to draw a tangent through the point to some similar and concentric conic, and the greater the given area, the greater will be the distance between the two conics. The area will therefore evidently be least when this last conic passes through the given point; and since the tangent at the point must be bisected, the line through a given point which cuts off the minimum area is bisected at that point.

In like manner, the chord drawn through a given point which cuts off the minimum or maximum area from any curve is bisected at that point. In like manner can be proved the following two theorems. I am indebted to the late Professor Mac Cullagh for my knowledge of all the theorems of this article, and I do not remember having seen them elsewhere published.

Ex. 1. *If a tangent* AB *to one curve cut off a constant arc from another, it is divided at the point of contact, so that* AP : PB *inversely as the tangents to the outer curve at* A *and* B.

Ex. 2. *If the tangent* AB *be of a constant length, and if the per-*

pendicular let fall on AB *from the intersection of the tangents at* A *and* B *meet* AB *in* M, *then* AP *will* = MB.

334. *To find the radius of curvature at any point on an ellipse.*

The centre of the circle circumscribing any triangle is the intersection of perpendiculars erected at the middle points of the sides of that triangle; it follows, therefore, that the centre of the circle passing through three consecutive points on the curve is the intersection of two consecutive normals to the curve.

Now, given any two triangles FPF′, FP′F′, and PN, P′N, the two bisectors of their vertical angles, it is easily proved, by elementary geometry, that twice the angle PNP′ = PFP′ + PF′P′. (See figure, p. 290).

Now, since the arch of any circle is proportional to the angle it subtends at the centre (Euc. VI. 33), and also to the radius (Art. 328), if we consider PP′ as the arch of a circle, whose centre is N, the angle PNP′ is measured by $\frac{PP'}{PN}$. In like manner, taking FR = FP, PFP′ is measured by $\frac{PR}{FP}$, and we have

$$\frac{2PP'}{PN} = \frac{PR}{FP} + \frac{P'R'}{F'P'};$$

but $$PR = P'R' = PP' \sin PP'F;$$

therefore, denoting this angle by θ, PN by R, FP, F′P, by ρ, ρ', we have

$$\frac{2}{R \sin\theta} = \frac{1}{\rho} + \frac{1}{\rho'}.$$

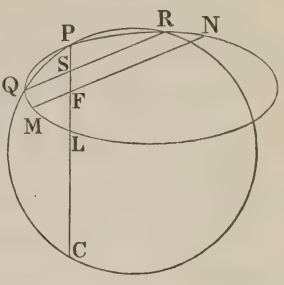
Hence it may be inferred that *the focal chord of curvature is double the harmonic mean between the focal radii.* Substituting $\frac{b}{b'}$ for $\sin\theta$, $2a$ for $\rho + \rho'$, and b'^2 for $\rho\rho$, we obtain the known value,

$$R = \frac{b'^3}{ab}.$$

The radius of curvature of the hyperbola or parabola can be investigated by an exactly similar process. In the case of the parabola we have ρ' infinite, and the formula becomes

$$\frac{2}{R \sin\theta} = \frac{1}{\rho}.$$

I am indebted to Mr. Townsend for the following investigation, by a different method, of the length of the focal chord of curvature:



Draw *any* parallel QR to the tangent at P, and describe a circle through PQR meeting the focal chord PL of the conic at C. Then by the circle PS.SC = QS.SR, and by the conic (Art. 194)

$$PS.SL : QS.SR :: PL : MN;$$

therefore, whatever be the circle,

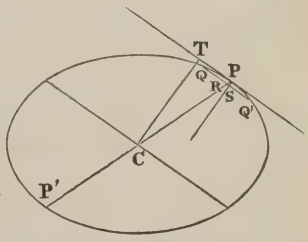
$$SC : SL :: MN : PL;$$

but for the circle of curvature the points S and P coincide, therefore

$$PC : PL :: MN : PL;$$

or *the focal chord of curvature is equal to the focal chord of the conic drawn parallel to the tangent at the point* (p. 209).

335. The radius of curvature of a central conic may otherwise be found thus:



Let Q be an indefinitely near point on the curve, QR a parallel to the tangent, meeting the normal in S; now, if a circle be described passing through P, Q, and touching PT at P, since QS is a perpendicular let fall from Q on the diameter of this circle, we have PQ^2 = PS multiplied by the diameter; or the radius of curvature = $\frac{PQ^2}{2PS}$. Now, since QR is always drawn parallel to the tangent, and since PQ must ultimately coincide with the tangent, we have PQ ultimately equal to QR; but, by the property of the ellipse (if we denote CP and its conjugate by a', b'),

$$b'^2 : a'^2 :: QR^2 : PR.RP' (= 2a'.PR),$$

therefore

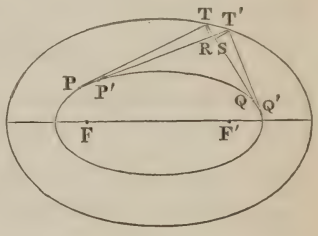
$$QR^2 = \frac{2b'^2.PR}{a'}.$$

Hence the radius of curvature = $\frac{b'^2}{a'} \cdot \frac{PR}{PS}$. Now, no matter how

small PR, PS, are taken, we have, by similar triangles, their *ratio* $\frac{PR}{PS} = \frac{CP}{CT} = \frac{a'}{p}$. Hence radius of curvature $= \frac{b'^2}{p}$.

It is not difficult to prove that *at the intersection of two confocal conics the centre of curvature of either is the pole of its own tangent with respect to the other.*

336. *If two tangents be drawn to an ellipse from any point of a confocal ellipse, the excess of the sum of these two tangents over the arc intercepted between them is constant.**



For, take an indefinitely near point T′, and let fall the perpendiculars TR, T′S, then (Art. 328)

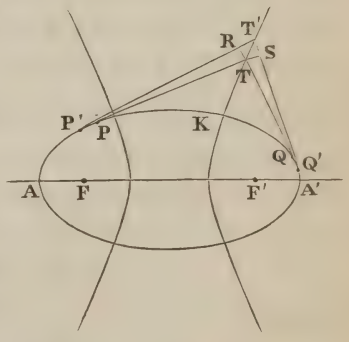
$$PT = PR = PP' + P'R$$

(for P′R may be considered as the continuation of the line PP′); in like manner, $Q'T' = QQ' + QS$.

Again, since, by Art. 189, the angle $TT'R = T'TS$, we have $TS = T'R$. If, therefore, in the quantity $TP + TQ - PQ$ we substitute for TP, PP′ + P′R, for TQ, TS + T′Q′ − QQ′, for PQ, PP′ + P′Q, we find this quantity $= T'P' + T'Q' - P'Q'$.

Cor. The same theorem will be true of any two curves which possess the property that two tangents, TP, TQ, to the inner one, always make equal angles with the tangent TT′ to the outer.

337. *If two tangents be drawn to an ellipse from any point of a confocal hyperbola, the difference of the arcs* PK, QK, *is equal to the difference of the tangents* TP, TQ.†



For it appears, precisely as before, that the excess of T′P′ − P′K over TP − PK = T′R, and that the excess of T′Q′ − Q′K over TQ − QK is T′S, which is equal

* This beautiful theorem was discovered by Mr. Graves. See his *Translation of Chasles's Memoirs on Cones and Spherical Conics*, p. 77.

† This extension of the preceding theorem was discovered by Mr. Mac Cullagh. *Dublin*

to T'R, since (Art. 189) TT' bisects the angle RT'S. The difference, therefore, between the excess of TP over PK, and that of TQ over QK, is constant; but in the particular case where T coincides with K, both these excesses, and consequently their difference, vanish; in every case, therefore, TP – PK = TQ – QK.

Cor. 1. *Fagnani's theorem*, "that an elliptic quadrant can be so divided, that the difference of its parts may be equal to the difference of the semiaxes," follows immediately from this Article, since we have only to draw tangents at the extremities of the axes, and through their intersection to draw a hyperbola confocal with the given ellipse. The co-ordinates of the points where it meets the ellipse are found to be

$$x^2 = \frac{a^3}{a+b}, \quad y^2 = \frac{b^3}{a+b}.$$

Cor. 2. It may also be inferred from this Article, that if a quadrilateral circumscribe an ellipse, two of whose vertices rest on a confocal conic, each other pair of vertices must lie on a confocal conic. Otherwise we should be able to find a right line equal in length to an elliptic arc. This theorem, however, can easily be proved directly.*

THE METHOD OF PROJECTIONS.†

338. We have already several times had occasion to point out to the reader the advantage gained by taking notice of the number of particular theorems often included under one general enunciation, but we now propose to lay before him a short sketch of a method which renders us a still more important service, and which enables us to tell when from a particular given theorem

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Exam. Papers, 1841, p. 41; 1842, pp. 68, 83. M. Chasles afterwards independently noticed the same extension of Mr. Graves's theorem. *Comptes rendus*, October, 1843, tom. xvii. p. 838.

* The reader will find some beautiful applications of the method used in this section to the rectification of conic sections, in a paper by Mr. Mac Cullagh, published in vol. xvi. of the *Transactions of the Royal Irish Academy*.

† This method is the invention of M. Poncelet. See his *Traité des Propriétés Projectives*, published in the year 1822. I shall be glad if the slight sketch here given induces any reader to study a work, from which I have perhaps derived more information than from any other on the theory of curves.

we can safely infer the general one under which it is contained. The method of projections, indeed, as requiring some knowledge of the geometry of three dimensions, may seem scarcely admissible into a treatise on *plane* geometry; yet, as it is only in laying down the principles of the method that we shall have to use a few not very difficult theorems of solid geometry, and as the applications of the method (the principles being once admitted) do not require any consideration of space of three dimensions, we feel that this beautiful method could not with propriety be excluded from the present treatise. The reader will have less difficulty in following the demonstrations here given, as in studying spherical trigonometry he has been already introduced to the consideration of space of three dimensions (*Luby's Trig.* p. 57).

339. If all the points of any figure be joined to any fixed point in space (O), the joining lines will form a *cone*, of which the point O is called the *vertex*, and the section of this cone, by any plane, will form a figure which is called the *projection* of the given figure. The plane by which the cone is cut is called the *plane of projection.*

To any point of one figure will correspond *a point in the other.*

For, if any point A be joined to the vertex O, the point *a*, in which the joining line OA is cut by any plane, will be the projection on that plane of the given point A.

A right line will always be projected into a right line.

For, if all the points of the right line be joined to the vertex, the joining lines will form a plane, and this plane will be intersected by any plane of projection in a right line.

Hence, if any number of points in one figure lie in a right line, the corresponding points on the projection also lie in a right line, and if any number of lines in one figure pass through a point, the corresponding lines on the projection will pass through a point.

340. *Any plane curve will always be projected into another curve of the same degree.*

For it is plain that, if the given curve be cut by any right line in any number of points, ABCD, &c., the projections will be cut

by the projection of that right line in *the same number* of corresponding points, *a*, *b*, *c*, *d*, &c., but (p. 209) the degree of a curve is estimated geometrically by the number of points in which it can be cut by any right line. If the line AB meet the curve in some real and some imaginary points, the line *ab* will meet the projection in the same number of real and the same number of imaginary points.

In like manner, if any two curves intersect, their projections will intersect in the same number of points, and any point common to one pair, whether real or imaginary, must be considered as the projection of a corresponding real or imaginary point common to the other pair.

Any tangent to one curve will be projected into a tangent to the other.

For, any line AB on one curve must be projected into the line *ab* joining the corresponding points of the projection. Now, if the points A, B, coincide, the points *a*, *b*, will also coincide, and the line *ab* will be a tangent.

More generally, if any two curves touch each other in any number of points, their projections will touch each other in the same number of points.

341. If a plane through the vertex parallel to the plane of projection meet the original plane in a line AB, then any pencil of lines diverging from a point on that line AB will be projected into a system of parallel lines on the plane of projection. For, since the line from the vertex to any point of AB meets the plane of projection at an infinite distance, the intersection of any two lines which meet on AB is projected to an infinite distance on the plane of projection. Conversely, *any system of parallel lines on the original plane is projected into a system of lines meeting on a point in the line* DF, *where a plane through the vertex parallel to the original plane is cut by the plane of projection.* The method of projections then leads us naturally to the conclusion, that any system of parallel lines may be considered as passing through a point at an infinite distance, for their projections on any plane pass through a point in general at a finite distance; and again, that *all the points at infinity on any plane may be considered as lying*

on a right line, since we have showed, that the projection of any point in which parallel lines intersect must lie somewhere on the right line DF in the plane of projection.

342. We see now that if any property of a given curve does not involve the *magnitude* of lines or angles, but merely relates to the *position* of lines as drawn to certain points, or touching certain curves, or to the position of points, &c., then this property will be true for any curve into which the given curve can be projected. Thus, for instance, "if through any point in the plane of a circle a chord be drawn, the tangents at its extremities will meet on a fixed line." Now since we shall presently prove that there is no curve of the second degree which cannot be projected into a circle, the method of projections shows at once that the properties of poles and polars are true not only for the circle, but also for all curves of the second degree. Again, Pascal's and Brianchon's theorems are properties of the same class, which it is sufficient to prove in the case of the circle, in order to know that they are true for all conic sections.

343. Properties which, if true for any figure, are true for its projection, are called *projective properties*. Beside the classes of theorems mentioned in the last Article, there are many projective theorems which *do* involve the magnitude of lines. For instance, the anharmonic ratio of four points in a right line, {ABCD} being measured by the ratio of the pencil {O.ABCD} drawn to the vertex, must be the same as that of the four points {*abcd*}, where this pencil is cut by any transversal. Again, if there be an equation between the mutual distances of any number of points in a right line, such as

$$AB.CD.EF + k.AC.BE.DF + l.AD.CE.BF + \&c. = 0,$$

where in each term of the equation the same points are mentioned, although in different orders, this property will be projective. For (see Art. 303) if for AB we substitute

$$\frac{OA.OB.\sin AOB}{OP}, \quad \&c.$$

each term of the equation will be divisible by

$$\frac{OA.OB.OC.OD.OE.OF}{OP^3},$$

and there will remain merely a relation between the sines of angles subtended at O. It is evident that the points ABCDEF need not be on the same right line; or, in other words, that the perpendicular OP need not be the same for all, provided the points be so taken that after the substitution, each term of the equation may be divisible by the same product $\frac{1}{OP.OP'.OP''}$, &c. Thus, for example, "If lines meeting in a point drawn through the vertices of a triangle ABC meet the opposite sides in the points *abc*, then $Ab.Bc.Ca = Ac.Ba.Cb$." This is a relation of the class just mentioned, and which it is sufficient to prove for any projection of the triangle ABC. Let us suppose the point C projected to an infinite distance, then AC, BC, C*c* are parallel, and the relation becomes $Ab.Bc = Ac.Ba$, the truth of which is at once perceived on making the figure.

344. It appears from what has been said, that if we wish to demonstrate any projective property of any figure, it is sufficient to demonstrate it for the *simplest* figure into which the given figure can be projected; *e. g.* for one in which any line of the given figure is at an infinite distance.

Thus, if it were required to investigate the harmonic properties of a complete quadrilateral ABCD, whose opposite sides intersect in E, F, and the intersection of whose diagonals is G, we may join all the points of this figure to any point in space O, and cut the joining lines by any plane parallel to OEF, then EF is projected to infinity, and we have a new quadrilateral, whose sides *ab*, *cd* intersect at *e* at infinity, that is, are parallel; while *ad*, *bc* intersect in a point *f* at infinity, or are also parallel. We thus see that *any quadrilateral may be projected into a parallelogram.* Now since the diagonals of a parallelogram bisect each other, the diagonal *ac* is cut harmonically in the points *a*, *g*, *c*, and the point where it meets the line at infinity *ef*. Hence AB is cut harmonically in the points A, G, C, and where it meets EF.

Ex. 2. *If two triangles* ABC, A′B′C′, *be such that the points of*

intersection of AB, A′B′; BC, B′C′; CA, C′A′; *lie in a right line, then the lines* AA′, BB′, CC′ *meet in a point.*

Project to infinity the line in which AB, A′B′, &c., intersect; then the theorem becomes: "If two triangles *abc*, *a′b′c′* have the sides of the one respectively parallel to the sides of the other, then the lines *aa′*, *bb′*, *cc′* meet in a point." But the truth of this latter theorem is evident, since *aa′*, *bb′* both cut *cc′* in the same ratio.

345. Before giving examples of the application of the method of projections to curves of the second degree, we wish to examine more particularly than in Art. 340 the nature of the section made by any plane in a cone standing on a circular base. We there proved that the projection of a circle must be always a curve of the second degree, and we wish now to show that the same circle may be projected into any of the three species of curves of the second degree. We commence by proving that *any curve will be projected into a similar curve, on a plane parallel to the plane of the original curve.*

For take any fixed point A in the plane of one of them, and the corresponding point *a* in the plane of the other, and let radii vectores be drawn from them to any corresponding points B, *b*; now from the similar triangles OAB, O*ab*, AB is to *ab* in the constant ratio OA : O*a*; and since *every* radius vector of the one curve is in this constant ratio to the corresponding radius vector of the other, the two curves are similar (Art. 238).

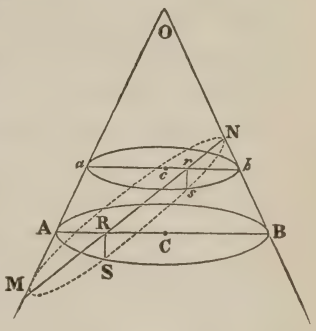
Cor. If a cone standing on a circular base be cut by any plane parallel to the base, the section will be a circle. This is evident as before: we may, if we please, suppose the points A, *a* the centres of the curves.

346. *The section by any plane of a cone standing on a circular base is a curve of the second degree.*

A cone of the second degree is said to be *right* if the line joining the vertex to the centre of the circle which is taken for base be perpendicular to the plane of that circle; in which case this line is called the *axis* of the cone. If this line be not per-

pendicular to the plane of the base, the cone is said to be *oblique*. The investigation of the sections of an oblique cone is exactly the same as that of the sections of a right cone, but we shall treat them separately, because the figure in the latter case being more simple will be more easily understood by the learner, who may at first find some difficulty in the conception of figures in space.

Let a plane (OAB) be drawn through the axis of the cone OC perpendicular to the plane of the section, so that both the section MS*s*N and also the base ASB are supposed to be perpendicular to the plane of the paper: the line RS, in which the section meets the base, is, therefore, also supposed perpendicular to the plane of the paper. Let the section cut the plane OAB in the line MN, which we shall first suppose to meet both the sides OA, OB, as in the figure, on the same side of the vertex.



Now let a plane parallel to the base be drawn at any other point *s* of the section. Then we have (Euc. III. 35) the square of RS, the ordinate of the circle, = AR.RB, and in like manner $rs^2 = ar.rb$. But from a comparison of the similar triangles ARM, *ar*M; BRN, *br*N, it can at once be proved that

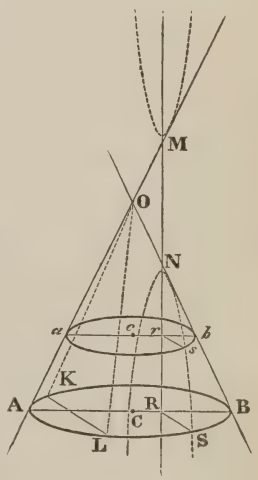
AR.RB : MR.RN :: *ar*.*rb* : M*r*.*r*N.

Therefore

$$RS^2 : rs^2 :: MR.RN : Mr.rN.$$

Hence the section MS*s*N is such that the square of any ordinate *rs* is to the rectangle under the parts in which it cuts the line MN in the constant ratio $RS^2 : MR.RN$. Hence it can immediately be inferred (Art. 147) that the section is an *ellipse*, of which MN is the axis major, while the square of the axis minor is to MN^2 in the given ratio

$$RS^2 : MR.RN.$$

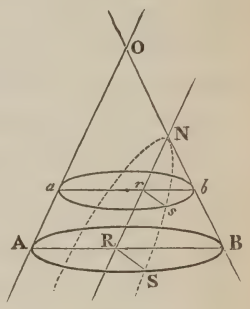


Secondly. Let MN meet one of the sides OR *produced*. The proof proceeds exactly as before, only that now we prove the square of the ordinate rs in a constant ratio to the rectangle $Mr.rN$ under the parts into which it cuts the line MN *produced*. The learner will have no difficulty in proving that the locus will in this case be a *hyperbola*, consisting evidently of the two opposite branches $N_s S$, $M_{s'} S'$.

Thirdly. Let the line MN be *parallel* to one of the sides. In this case, since $AR = ar$, and $RB : rb :: RN : rN$, we have the square of the ordinate $rs\,(= ar.rb)$ to the abscissa rN in the constant ratio

$$RS^2\,(= AR.RB) : RN.$$

The section is therefore a *parabola*.*



347. It is evident that the projections of the tangents at the points A, B of the circle are the tangents at the points M, N of the conic section (Art. 341); now in the case of the parabola the point M and the tangent at it go off to infinity; we are therefore again led to the conclusion that *every parabola has one tangent altogether at an infinite distance.*

348. Let the cone now be supposed oblique. The plane of the

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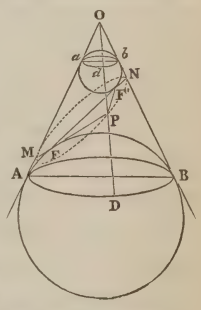
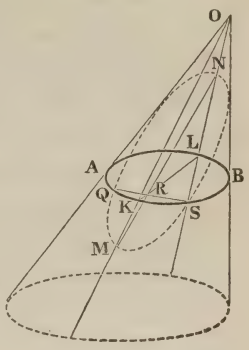
* It is worth mentioning, that *if a sphere be inscribed in a right cone touching the plane of any section, the point of contact will be a focus of that section, and the corresponding directrix will be the intersection of the plane of the section with the plane of contact of the cone with the sphere.*



Let a sphere be both inscribed and exscribed between the cone and the plane of the section. Now, if any point P of the section be joined to the vertex, and the joining line meet the planes of contact in Dd, then we have $PD = PF$, since they are tangents to the same sphere, and, similarly, $Pd = PF'$, therefore $PF + PF' = Dd$, which is constant. The point (R) where FF' meets AB produced, is a point on the directrix, for by the property of the circle NFMR is cut harmonically, therefore, R is a point on the polar of F. This is only a particular case of a more general theorem.

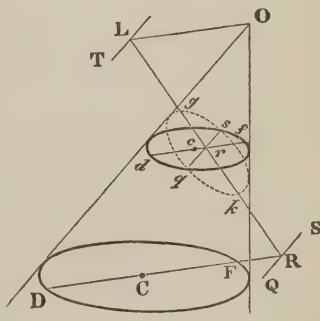
The reader will have no difficulty in proving that the parameter of the section MPN is constant, if the distance of the plane from the vertex be constant.

paper is a plane drawn through the line OC, perpendicular to the plane of the circle AQSB. Now let the section meet the base in any line QS, draw a diameter LK bisecting QS, and let the section meet the plane OLK in the line MN, then the proof proceeds exactly as before; we have the square of the ordinate RS equal to the rectangle LR.RK; if we conceive a plane, as before, drawn parallel to the base (which, however, is left out of the figure in order to avoid rendering it too complicated), we have the square of any other ordinate, *rs*, equal to the corresponding rectangle *lr*.*rk*; and we then prove by the similar triangles KRM, *kr*M; LRN, *lr*N, in the plane OLK, exactly as in the case of the right cone, that $RS^2 : rs^2$, as the rectangle under the parts in which each ordinate divides MN, and that therefore the section is a conic of which MN is the diameter bisecting QS, and which is an ellipse when MN meets both the lines OL, OK on the same side of the vertex, an hyperbola when it meets them on different sides of the vertex, and a parabola when it is parallel to either.



In the proof just given QS is supposed to intersect the circle in real points; if it did not, we have only to take, instead of the circle AB, any other parallel circle *ab*, which *does* meet the section in real points, and the proof will proceed as before.

349. *If a circular section be cut by any plane in a line* RS *the rectangle* DR.RF *of the segments of the diameter of the circle conjugate to* QS *is to the rectangle* gR.Rk *under the segments of the diameter of the section conjugate to* QS, *as the square of the diameter of the section parallel to* QS *is to the square of the conjugate diameter* gk.



This has been proved in the last Article, in the case where QS meets the circle in real points, since $rs^2 = dr.rf$. Now, if the

plane meet any other parallel plane in a line QS which does not meet the curve: First, we say that the diameters conjugate to QS with regard to the circle, and with regard to the other section, will meet QS in the same point R, for it is evident, by Art. 345, that the diameter *df*, bisecting chords of any circular section parallel to *qs*, will be projected into a diameter bisecting the parallel chords of any parallel section. The middle points, therefore, of all chords parallel to *qs*, must lie in the plane O*df*, and, consequently, the diameter conjugate to QS, in the section *gqks*, must be the line *gk*, in which it is met by the plane O*df*. DF, therefore, and *gk*, intersect in the point R, where QS meets the plane O*df*.

Now, since we have proved that the lines *gk*, *df*, DF, lie in one plane passing through the vertex, the points D, *d*, are projections of *g*, that is, they lie in one right line passing through the vertex; we have, therefore, by similar triangles, as in Art. 346, $dr.rf : \mathrm{DR.RF} :: gr.rk : g\mathrm{R}.\mathrm{R}k$; and, since $dr.rf : gr.rk$, as the squares of the parallel semidiameters, $\mathrm{DR.RF} : g\mathrm{R}.\mathrm{R}k$ in the same ratio.

350. If a plane be drawn through the vertex parallel to the circular base meeting the section *gqks* in TL, it follows, as a particular case of the preceding, that $g\mathrm{L}.\mathrm{L}k : \mathrm{OL}^2$ in the ratio of the squares of the parallel diameters of the section. Hence we see that, given any conic section and a line, TL, in its plane, it is an indeterminate problem to find O the vertex of a cone such that the section of it, by any plane parallel to OTL, should be a circle. For, draw the diameter of the section conjugate to TL, then the distance of L from the vertex of the cone is determined by the present Article; also OL must lie in the plane perpendicular to TL, since it is parallel to the diameter of a circle perpendicular to TL; O may, therefore, be any point of a certain circle in a plane perpendicular to TL.

Hence, *given any conic section, and any line* TL *in its plane not cutting it, we can project it so that the conic section may become a circle; and the line may be projected to infinity*, for we have only to take any point O, such that the plane OTL may be parallel to the planes of circular section, and then any plane parallel to OTL will be a plane of projection fulfilling the required conditions.

351. *Given any conic section and a point in its plane, we can project it into a circle, of which the projection of that point is the centre,* for we have only to project it so that the projection of the polar of the given point may pass to infinity (Art. 152).

Or again, *Any two conic sections may be projected so as both to become circles,* for we have only to project one of them into a circle so as that any of its chords of intersection with the other shall pass to infinity, and then, by Art. 253, the projection of the second conic passing through the same points at infinity as the circle must be a circle also.

Any two conics which have double contact with each other may be projected into concentric circles.

For we have only to project one of them into a circle so that its chord of contact with the other may pass to infinity (Art. 253).

Strictly speaking, all these projections have only been shown to be possible when the line projected to infinity does not meet the conic in real points, but this is a limitation which it is unnecessary in practice to attend to. The reader is, probably, already convinced that the distinction between real and imaginary, however much it affects the shape and outward appearance of curves, has comparatively little influence on their projective properties; and that a projective proposition once proved true for any state of a figure *may* become *unmeaning*, but will *never* become *false*, when certain lines in that figure have become imaginary. Thus, for example, although the method of projecting into concentric circles only directly proves properties of conics having double contact, *whose chord of contact is imaginary*, we shall not think it necessary to seek for an independent proof of the same properties in the case where the chord of contact is real.

352. We shall now give some examples of the method of deriving properties of conics from those of the circle, or from other more particular properties of conics.

Ex. 1. "A line through any point is cut harmonically by the curve and the polar of that point." This property and its reciprocal are projective properties (Art. 343), and both being true for the circle are true for every conic. Hence all the properties of the circle depending on the theory of poles and polars, are true for all the conic sections.

Ex. 2. The *anharmonic* properties of the points and tangents of a conic are projective properties, which, when proved for the circle, as in Art. 302, are proved for all the conics. Hence, every property of the circle which results from either of its anharmonic properties is true also for all the conic sections.

Ex. 3. Carnot's theorem (Art. 303), that if a conic meet the sides of a triangle,

$$Ab.Ab'.Bc.Bc'.Ca.Ca' = Ac.Ac'.Ba.Ba'.Cb.Cb',$$

is a projective property which need only be proved in the case of the circle, in which case it is evidently true, since

$$Ab.Ab' = Ac.Ac', \text{ \&c.}$$

The theorem is evidently true, and can be proved in like manner for any polygon.

Ex. 4. From Carnot's theorem, thus proved, could be deduced the properties of Art. 146, by supposing the point C at an infinite distance; we then have

$$\frac{Ab.Ab'}{Ac.Ac'} = \frac{Ba.Ba'}{Bc.Bc'},$$

where the line Ab is parallel to Ba.

Ex. 5. Given two concentric circles, any chord of one which touches the other is bisected at the point of contact.

Given two conics having double contact with each other, any chord of one which touches the other is cut harmonically at the point of contact, and where it meets the chord of contact of the conics. (Art. 321.)

For the line at infinity in the first case is projected into the chord of contact of two conics having double contact with each other. Ex. 4, Art. 242, is only a particular case of this theorem.

Ex. 6. Given three concentric circles, any tangent to one is cut by the other two in four points whose anharmonic ratio is constant.

Given three conics all touching each other in the same two points, any tangent to one is cut by the other two in four points whose anharmonic ratio is constant.

The first theorem is obviously true, since the four lengths are constant. The second may be considered as an extension of the anharmonic property of the tangents of a conic. In like manner, the theorems (in Art. 275) with regard to anharmonic ratios in

conics having double contact are immediately proved by projecting the conics into concentric circles.

Ex. 7. We mentioned already, that it was sufficient to prove Pascal's theorem for the case of a circle, but by the help of Art. 341 we may still further simplify our figure, for we may suppose the line joining the intersection of AB, DE, to that of BC, EF, to pass off to infinity; and it is only necessary to prove that, if a hexagon be inscribed in a circle having the side AB parallel to DE, and BC to EF, then CD will be parallel to AF, but the truth of this can be shown from elementary considerations.

Ex. 8. *A triangle is inscribed in any conic, two of whose sides pass through fixed points, to find the envelope of the third* (p. 233). Let the line joining the fixed points be projected to infinity, and at the same time the conic into a circle, and this property becomes, "A triangle is inscribed in a circle, two of whose sides are parallel to fixed lines, to find the envelope of the third." But this envelope is a concentric circle, since the vertical angle of the triangle is given; hence, in the general case, the envelope is a conic touching the given conic in two points on the line joining the two given points.

Ex. 9. *To investigate the projective properties of a quadrilateral inscribed in a conic.* Let the conic be projected into a circle, and the quadrilateral into a parallelogram (Art. 344). Now the intersection of the diagonals of a parallelogram inscribed in a circle is the centre of the circle; hence the intersection of the diagonals of a quadrilateral inscribed in a conic is the pole of the line joining the intersections of the opposite sides. Again, if tangents to the circle be drawn at the vertices of this parallelogram, the diagonals of the quadrilateral so formed will also pass through the centre, bisecting the angles between the first diagonals; hence, "the diagonals of the inscribed and corresponding circumscribing quadrilateral pass through a point, and form an harmonic pencil."

Ex. 10. Given four points on a conic, the locus of its centre is a conic through the middle points of the sides of the given quadrilateral.

Given four points on a conic, the locus of the pole of any fixed line is a conic passing through the fourth harmonic to the point in which this line meets each side of the given quatrilateral.

Ex. 11. The locus of the point where parallel chords of a circle are cut in a given ratio, is an ellipse having double contact with the circle. (Art. 157.)

If through a fixed point O a line be drawn meeting the conic in A, B, and on it a point P be taken, such that {OABP} may be constant, the locus of P is a conic having double contact with the given conic.

Ex. 12. If from the vertex of an ellipse any radius vector be drawn, and a parallel radius vector through the centre, the locus of the intersection of this parallel, with a tangent through the variable point, is the tangent at the other vertex. (Art. 229.)

Given a fixed point P and its polar TT', and let any line through P meet the conic in a, a', then if a line PQ be drawn to the point where any radius vector aR meets TT', the locus of the intersection of PQ with the tangent at R will be the tangent at a'.

353. We may project several properties relating to foci by the help of the definition of a focus given page 237.

Ex. 1. The locus of the centre of a circle touching two given circles is a hyperbola, having the centres of the given circles for foci.

If a conic be described through two fixed points, and touching two conics which also pass through those points, the locus of the pole of the line joining those points is a conic inscribed in the quadrilateral formed by joining the two given points to the poles of the same line with regard to the given conics.

We give this example as worth the learner's study, because it illustrates the different principles that all circles pass through two fixed points at infinity (Art. 253); that the centre is the pole of the line joining them (Art. 152);* that a focus is the intersection of tangents passing through these fixed points (Art. 277); and that we are safe in extending our conclusion from imaginary to real points (Art. 351).

Ex. 2. Given the focus and two points of a conic section, the intersection of tangents at those points will be on a fixed line. (Art. 233.)

Given two tangents, and two points on a conic, the locus of the intersection of tangents at those points is a right line.

* We might have mentioned before, that it may be inferred that the polar of the centre of a conic is at an infinite distance, from the definition of the polar, page 135, and from the fact that tangents at the extremities of any chord passing through the centre are parallel.

Ex. 3. Given a focus and two tangents to a conic, the locus of the other focus is a right line. (This follows from Art. 189.)

Given four tangents to a conic, and a fixed point on each of two of them, the locus of the intersection of tangents from these points is a right line.

For, the two points at infinity on any circle lie one on each of the tangents from one focus, and the intersection of the other tangents from these two points is the other focus.

Ex. 4. Given three tangents to a parabola, the locus of the focus is the circumscribing circle. (Art. 224.)

Given four tangents to a conic, and two fixed points on any one of them, the locus of the intersection of the other tangents from these points is a conic passing through these two points, and circumscribing the triangle formed by the other three tangents.

For every parabola has one tangent at infinity, and the two points through which every circle must pass lie on this tangent.

Ex. 5. The locus of the centre of a circle passing through a fixed point, and touching a fixed line, is a parabola of which the fixed point is the focus.

Given one tangent, and three points on a conic, the locus to the intersection of tangents at any two of these points is a conic inscribed in the triangle formed by those points.

Ex. 6. Given four tangents to a conic, the locus of the centre is the line joining the middle points of the diagonals of the quadrilateral.

Given four tangents to a conic, the locus of the pole of any line, is the line joining the fourth harmonics of the points where the given line meets the diagonals of the quadrilateral.

It follows from our definition of a focus, that if two conics have the same focus, this point will be an intersection of common tangents to them, and will possess the properties mentioned in Art. 264. Also, that if two conics have the same focus and the same directrix they may be considered as two conics having double contact with each other, and may be projected into two concentric circles.

354. If we are given any property relating to the magnitude of angles, since angles which are constant in any figure will in general not be constant in the projection of that figure, we proceed to show what property of a projected figure may be inferred

from the given property,* and we commence with the case of the right angle.

Let the equations of two lines at right angles to each other be $x = 0$, $y = 0$, then the equation which determines the direction of the points at infinity on any circle is $x^2 + y^2 = 0$, or

$$x + y\sqrt{-1} = 0, \qquad x - y\sqrt{-1} = 0.$$

Hence (Art. 54) these four lines form an harmonic pencil. Hence, given four points, ABCD, of a line cut harmonically, where A, C may be real or imaginary, if these points be transferred by a real or imaginary projection, so that A, C may become the two imaginary points at infinity on any circle, then any lines through B, D, will be projected into lines at right angles to each other. Conversely, *any two lines at right angles to each other will be projected into lines which cut harmonically the line joining the two fixed points which are the projections of the imaginary points at infinity on a circle.* The following examples will illustrate the use of this principle :

Ex. 1. The tangent to a circle is at right angles to the radius.

Any chord of a conic is cut harmonically by any tangent, and by the line joining the point of contact of that tangent to the pole of the given chord. (Art. 142.)

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* M. Poncelet has only treated of some particular cases where constant angles are projected into constant angles. He has thus very ingeniously connected properties relating to angles subtended at the foci of conics with properties of the circle. These properties, however, follow so naturally from the method of reciprocal polars, that I have not thought it worth while to give up the space necessary to explain M. Poncelet's method of obtaining them by projection. Properties relating to angles in conics may also be reduced to properties relating to angles in circles by the help of the following principle, due to Mr. Mulcahy, which the reader will find easy to prove : " If, through any point C of any circular section of an oblique cone a plane be drawn making with the line to the vertex OC the same angle as that made by the plane of the circular section, and passing through the line perpendicular to OC in the plane of the circular section, then any two lines passing through C in the one plane will be projected into lines in the other plane making the same angle with each other." When C is the centre of the circular section it will be a focus of the other section. (*Chasles's Apercu Historique*, p. 287). Thus, for instance, from the property that " any two conjugate diameters of a circle are at right angles to each other," we obtain the theorem, that any line through the focus is at right angles to the radius vector drawn to its pole ; we can show that the envelope of a chord which subtends a constant angle at any point of a conic is another conic, &c.

For the chord of the conic is supposed to be the projection of the line at infinity on the plane of the circle; the points where the chord meets the conic will be the projections of the imaginary points at infinity on the circle; and the pole of the chord will be the projection of the centre of the circle.

Ex. 2. Any right line drawn through the focus of a conic is at right angles to the line joining its pole to the focus. (Art. 192.)

Any right line through any point, the two tangents at that point, and the line joining the given point to the pole of the given line, form a harmonic pencil. (Art. 144.)

It is evident that the first of these properties is only a particular case of the second, if we recollect that the tangents from the focus are the lines joining the focus to the two imaginary points in any circle (Art. 277).

Ex. 3. Let us apply Ex. 6 of the last Article to determine the locus of the pole of a given line with regard to a system of confocal conics. Being given the two foci we are given a quadrilateral circumscribing the conic (Art. 277), one of the diagonals of this quadrilateral is the line joining the foci, therefore (Ex. 6) one point on the locus is the fourth harmonic to the point where the given line cuts the distance between the foci. Again, another diagonal is the line at infinity, and since the extremities of this diagonal are the points at infinity on a circle, by the present Article the locus is perpendicular to the given line. The locus is, therefore, completely determined.

Ex. 4. Two confocal conics cut each other at right angles.

If two conics be inscribed in the same quadrilateral, the two tangents at any of their points of intersection cut any diagonal of the circumscribing quadrilateral harmonically.

The last theorem is a particular case of the reciprocal of the first theorem in Art. 321.

Ex. 5. The locus of the intersection of two tangents to a central conic, which cut at right angles, is a circle.

If from any two points B, D, which cut a given line AC harmonically, tangents be drawn to a conic, the locus of their intersection O is a conic through the points A, C.

The last theorem may, by Art. 144, be stated otherwise thus:

"The locus of a point O, such that the line joining O to the pole of AO may pass through C, is a conic through A, C;" and the truth of it is evident directly, by taking four positions of the line, when we see, by Art. 323, that the anharmonic ratio of four lines, AO, is equal to that of four corresponding lines, CO.

Ex. 6. The locus of the intersection of tangents to a parabola, which cut at right angles, is the directrix.

If in the last example the line AC touch the given conic, the locus of O will be the right line joining the points of contact of tangents from A, C.

Ex. 7. If from any point on a conic two lines at right angles to each other be drawn, the chord joining their extremities passes through a fixed point. (Art. 234.)

If a harmonic pencil be drawn through any point on a conic, two legs of which are fixed, the chord joining the extremities of the other legs will pass through a fixed point.

In other words, given two points, a, c, on a conic, and $\{abcd\}$ an harmonic ratio, bd will pass through a fixed point, namely, the intersection of tangents at a, c. But the truth of this may be seen directly: for let the line ac meet bd in K, then since $\{a.abcd\}$ is a harmonic pencil, the tangent at a cuts bd in the fourth harmonic to K: but so likewise must the tangent at c, therefore these tangents meet bd in the same point. As a particular case of this theorem we have the following: "Through a fixed point on a conic two lines are drawn, making equal angles with a fixed line through it, the chord joining their extremities will pass through a fixed point."

355. *A system of pairs of right lines drawn through a point, every two of which make equal angles with a fixed line, cut the line at infinity in a system of points in involution, of which the two points at infinity on any circle form one pair of conjugate points.* For (see figure, p. 275) they evidently cut *any* right line in a system of points in involution, and the two points at infinity just mentioned belong to the system, since they are cut harmonically by the given internal and external bisector of every pair of right lines. These bisectors meet any right line in the foci of the system in involution.

The tangents from any point to a system of confocal conics make equal

The tangents from any point to a system of conics inscribed in the same

angles with two fixed lines. (Art. 189.)

quadrilateral cut any diagonal of that quadrilateral in a system of points in involution of which the two extremities of that diagonal are a pair of conjugate points. (Art. 320.)

356. *Two lines diverging from a fixed point, which contain a constant angle, cut the line joining the two points at infinity on a circle, so that the anharmonic ratio of the four points is constant.*

For the equation of two lines containing an angle θ being $x = 0$, $y = 0$, the direction of the points at infinity on any circle is determined by the equation

$$x^2 + y^2 + 2xy\cos\theta = 0,$$

and, separating this equation into factors, we see, by Art. 54, that the anharmonic ratio of the four lines is constant if θ be constant.

Ex. 1. "The angle contained in the same segment of a circle is constant." We see, by the present Article, that this is the form assumed by the anharmonic property of four points on a circle when two of them are at an infinite distance.

Ex. 2. The envelope of a chord of a conic which subtends a constant angle at the focus is another conic having the same focus and the same directrix.

Ex. 3. The locus of the intersection of tangents to a parabola which cut at a given angle is a hyperbola having the same focus and the same directrix.

Ex. 4. If from the focus of a conic a line be drawn making a given angle with any tangent, the locus of the point where it meets it is a circle.

If tangents through any point O meet the conic in T, T′, and there be taken on the conic two points A, B, such that $\{O.ATBT'\}$ is constant, the envelope of AB is a conic touching the given conic in the points T, T′.

If in Art. 354, Ex. 6, the points B, D be so taken that $\{ABCD\}$ is constant, the locus of O is a conic touching the given conic at the points of contact of tangents from A, C.

If a variable tangent to a conic meet two fixed tangents in T, T′, and a fixed line in M, and there be taken on it a point P, such that $\{PTMT'\}$ may be constant, the locus of P is a conic passing through the points where the fixed tangents meet the fixed line.

A particular case of this theorem is: "*The locus of the point where the intercept of a variable tangent between two fixed tangents is cut in a given ratio, is a hyperbola whose asymptotes are parallel*

to the fixed tangents. This again includes, as a particular case, Art. 199.

Ex. 5. If from a fixed point O, OP be drawn to a given circle, and the angle TPO be constant, the envelope of TP is a conic having O for its focus.

Given the anharmonic ratio of a pencil whose vertex moves along a given conic, and three of whose legs pass through fixed points, two of which are on the given conic, the envelope of the fourth leg is a conic touching the lines joining the fixed point not on the conic to the other two.

A particular case of this is: "If two fixed points A, B, on a conic be joined to a variable point P, and the intercept made by the joining chords on a fixed line be cut in a given ratio at M, the envelope of PM is a conic touching parallels through A and B to the fixed line."

Ex. 6. If from a fixed point O, OP be drawn to a given right line, and the angle TPO be constant, the envelope of TP is a parabola having O for its focus.

Given the anharmonic ratio of a pencil, three of whose legs pass through fixed points, and whose vertex moves along a fixed line, the envelope of the fourth leg is a conic touching the three sides of the triangle formed by the three given points.

357. We remarked already, that the advantage gained by the method of projections is very analogous to that gained by the use of trilinear co-ordinates. In fact, when we show that an equation expressed in terms of α, β, γ, is identical in form with an equation expressed in terms of x, y, k, it is the same as if we say that the line γ can be projected to infinity. When we show that the equation $LM = R^2$ can be treated by exactly the same methods as the equations $xy = k^2$ or $px = y^2$, it is the same as if we say that, "given two tangents to a conic, and their chord of contact, we can project the conic either into a hyperbola, of which the projections of the given tangents shall be asymptotes, or into a parabola, of which the projection of one of the given tangents shall be a diameter."*

* The method of projections can equally be used in obtaining from properties of plane curves properties of other curves not plane, *e. g.* curves on the surface of a sphere.

Mr. Mulcahy, some years ago, gave the following method of obtaining the properties

358. We shall conclude this chapter with a brief account of the method of *orthogonal projection*, which, before the publication of M. Poncelet's treatise, was the only method of projection much used by geometers. If from all the points of any figure perpendiculars be let fall on any plane, the feet of these perpendiculars will trace out a figure which is called the *orthogonal projection* of the given figure. The orthogonal projection of any figure is, therefore, a right section of a *cylinder* passing through the given figure.

All parallel lines are in a constant ratio to their orthogonal projections on any plane.

For (see fig. p. 4) MM′ represents the orthogonal projection of the line PQ, and it is evidently = PQ multiplied by the cosine of the angle which PQ makes with MM′.

All lines parallel to the intersection of the plane of the figure with the plane on which it is projected, are equal to their orthogonal projections.

For, since the intersection of the planes is itself not altered by projection, neither can any line parallel to it.

The area of any figure in a given plane is in a constant ratio to its orthogonal projection on another given plane.

For, if we suppose ordinates of the figure and of its projection

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of angles subtended at the focus from those of small circles on a sphere. The method depends on the following principle: *the locus of the vertices of all the right cones from which a given ellipse can be cut is a hyperbola passing through the foci of the ellipse.* For, see note, p. 304, the difference of MO and NO is constant, being equal to the difference of MF′ and NF′.

Now, let us take any property of a small circle of a sphere, *e. g.*, if through any point P, on the surface of a sphere, a great circle be drawn, cutting the small circle in the points A, B, then tan $\frac{1}{2}$AP tan $\frac{1}{2}$BP is constant. Now, let us take a cone whose base is the small circle, and whose vertex is the centre of the sphere, and let us cut this cone by any plane, and we learn that "if through a point p, in the plane of any conic, a line be drawn cutting the conic in the points a, b, then the product of the tangents of the halves of the angles which ap, bp subtend at the vertex of the cone will be constant; this property will be true of the vertex of any right cone, out of which the section can be cut, and, therefore, since the focus is a point in the locus of such vertices, it must be true that tan $\frac{1}{2}afp$ tan $\frac{1}{2}bfp$ is constant.

Mr. Mac Cullagh has derived this theorem, by a very elegant geometrical proof, from the fundamental property of the focus and directrix. The product of the tangents will remain fixed if p be any point on a conic having same focus and directrix.

to be drawn perpendicular to the intersection of the planes, since every ordinate of the projection is to the corresponding ordinate of the original figure in the constant ratio of the cosine of the angle between the planes to unity, by Art. 331, Cor., the areas of the figures will be in the same ratio.

Any ellipse can be orthogonally projected into a circle.

For, if we take the intersection of the plane of projection with the plane of the given ellipse parallel to the axis minor of that ellipse, and if we take the cosine of the angle between the planes $= \frac{b}{a}$, then every line parallel to the axis minor will be unaltered by projection, but every line parallel to the axis major will be shortened in the ratio $b : a$, the projection will, therefore (Art. 157), be a circle, whose radius is b.

359. We shall apply the principles laid down in the last Article to prove the following theorem: *The radius of the circle circumscribing any triangle inscribed in an ellipse* $= \frac{b'b''b'''}{ab}$, *where* b', b'', b''', *are the semidiameters of the ellipse parallel to the sides of the triangle.** The expression given already for the radius of curvature of any point on an ellipse is plainly only a particular case of the present theorem.

Let the sides of the triangle be α, β, γ, and its area A, then, by elementary geometry,

$$R = \frac{\alpha\beta\gamma}{4A}.$$

Now let the ellipse be projected into a circle whose radius is b, then, since this is the circle circumscribing the projected triangle, we have

$$b = \frac{\alpha'\beta'\gamma'}{4A'}.$$

But, since parallel lines are in a constant ratio to their projections, we have

$$\alpha' : \alpha :: b : b',$$
$$\beta' : \beta :: b : b'',$$
$$\gamma' : \gamma :: b : b''';$$

* I take this theorem from an Examination Paper given by Mr. Mac Cullagh in the year 1836; but the proof here given is due, I believe, to Mr. Graves.

and, since (Art. 358) A′ is to A as the area of the circle ($=\pi b^2$) to the area of the ellipse ($=\pi ab$), we have

$$A' : A :: b : a.$$

Hence

$$\frac{a'\beta'\gamma'}{4A'} : \frac{a\beta\gamma}{4A} :: ab^2 : b'b''b''',$$

and, therefore,

$$R = \frac{b'b''b'''}{ab}.$$

It can be proved, in general, that if c, c', c'', be chords drawn through the focus parallel to the sides of a triangle inscribed in *any conic*, and p the parameter of the conic,

$$R^2 = \frac{cc'c''}{4p}.*$$

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* Dublin Examination Papers, 1836, p. 22.

NOTES.

Art. 59, Page 53.

It would have been more symmetrical to have expressed the four sides $A = 0$, $B = 0$, $C = 0$, $D = 0$, these four equations being necessarily connected by an identical relation,

$$aA + bB + cC + dD = 0,$$

then $aA + bB = 0$, or $cC + dD = 0$, is the equation of the lines joining the points (AB), (CD); and so for the rest.

Pascal's Theorem, Page 223.

M. Steiner was the first who (in Gergonne's *Annales*) directed the attention of geometers to the complete figure obtained by joining in every possible way six points on a conic. M. Steiner's theorems were corrected and extended by M. Plücker (Crelle's Journal, vol. v. p. 274), and the subject has been more recently investigated by Messrs. Cayley and Kirkman, the latter of whom, in particular, has added several new theorems to those already known. We shall in this note give a slight sketch of the more important of these, and of the methods of obtaining them. The greater part are derived by joining the simplest principles of the theory of combinations with the following elementary theorems and their reciprocals: "If two triangles be such that the lines joining corresponding vertices meet in a point (which we shall call the *pole* of the two triangles), the intersections of corresponding sides will lie in one right line (which we shall call their *axis*)." "If the intersections of opposite sides of three triangles be for each pair *the same* three points in a right line, the poles of the first and second, second and third, third and first, will lie in a right line." Let the six points be a, b, c, d, e, f, which we shall call the points P. These may be connected by *fifteen* right lines, ab, ac, &c., which we shall call the lines C. Each of the

lines C (for example ab) is intersected by the fourteen others; by four of them in the point a, by four in the point b, and consequently by six in points distinct from the points P (for example the points ab, cd; &c.) These we shall call the points p. There are forty-five such points; for there are six on each of the lines C. To find then the number of points p, we must multiply the number of lines C by 6, and divide by 2, since two lines C pass through every point p.

If we take the sides of the hexagon in the order $abcdef$, Pascal's theorem is, that the three p points, (ab, de), (cd, fa), (bc, ef), lie in one right line, which we may call either the Pascal $abcdef$, or else we may denote as the Pascal $\left\{\begin{matrix} ab.cd.ef \\ de.fa.bc \end{matrix}\right\}$, a form which we sometimes prefer, as showing more readily the three points through which the Pascal passes. Through each point p four Pascals can be drawn. Thus through (ab, de) can be drawn $abcdef$, $abfdec$, $abcedf$, $abfedc$. We then find the total number of Pascals by multiplying the number of points p by 4, and dividing by 3, since there are three points p on each Pascal. We thus obtain the number of Pascal's lines = 60. We might have derived the same directly by considering the number of different ways of arranging the letters $abcdef$.

Consider now the three triangles whose sides are

$$ab,\ cd,\ ef, \quad (1)$$
$$de,\ fa,\ bc, \quad (2)$$
$$cf,\ be,\ ad, \quad (3)$$

The intersections of corresponding sides of 1 and 2 lie on the same Pascal, therefore the lines joining corresponding vertices meet in a point, but these are the three Pascals,

$$\left\{\begin{matrix} ab.de.cf \\ cd.fa.be \end{matrix}\right\}, \quad \left\{\begin{matrix} cd.fa.be \\ ef\ .bc\ .ad \end{matrix}\right\}, \quad \left\{\begin{matrix} ef.bc.ad \\ ab.de.cf \end{matrix}\right\}.$$

This is Steiner's theorem (p. 224); we shall call this the g point,

$$\left\{\begin{matrix} ab\ .\ de\ .\ cf \\ cd.\ fa\ .\ be \\ ef\ .\ bc\ .\ ad \end{matrix}\right\}.$$

The notation shows plainly that on each Pascal's line there is only one g point; for given the Pascal $\left\{\begin{matrix} ab.de.cf \\ cd.fa.be \end{matrix}\right\}$ the g point on it is found by writing under each term the two letters not already found in that vertical line. Since then three Pascals intersect in every point g, the number of points $g = 20$. If we take the triangles 2, 3; and 1, 3; the

lines joining corresponding vertices are the same in all cases: therefore, by the reciprocal of the second preliminary theorem, the three *axes* of the three triangles meet in a point. This, however, is plainly only the g point $\left\{\begin{matrix} ab.cd.ef \\ de.fa.bc \\ cf.be.ad \end{matrix}\right\}$, and therefore leads us to no new theorem.

Let us now consider the triangles,

$$\begin{matrix} ab & cd & ef & (1) \\ \left.\begin{matrix} ab.ce.df \\ de.bf.ac \end{matrix}\right\}, & \left.\begin{matrix} cd.bf.ae \\ af.ce.bd \end{matrix}\right\}, & \left.\begin{matrix} ef.bd.ac \\ bc.ae.df \end{matrix}\right\}, & (4) \\ \left.\begin{matrix} ab.ce.df \\ cf.bd.ae \end{matrix}\right\}, & \left.\begin{matrix} cd.bf.ae \\ be.ac.df \end{matrix}\right\}, & \left.\begin{matrix} ef.bd.ac \\ ad.ce.bf \end{matrix}\right\}, & (5) \end{matrix}$$

Now the intersections of corresponding sides of 1 and 4 are three points which lie on the same Pascal; therefore the lines joining corresponding vertices meet in a point. But these are the three Pascals,

$$\left.\begin{matrix} ab.ce.df \\ cd.bf.ae \end{matrix}\right\}, \quad \left.\begin{matrix} cd.bf.ae \\ ef.ac.bd \end{matrix}\right\}, \quad \left.\begin{matrix} ef.ac.bd \\ ab.df.ce \end{matrix}\right\}.$$

We may denote the point of meeting as the h point, $\left\{\begin{matrix} ab.ce.df \\ cd.bf.ae \\ ef.ac.bd \end{matrix}\right\}$.

The notation differs from that of the g points in that only one of the vertical columns contains the six letters without omission or repetition. On every Pascal there are three h points, viz., there are on

$$\left.\begin{matrix} ab.cd.ef \\ de.af.bc \end{matrix}\right\}; \quad \left\{\begin{matrix} \overline{ab}.cd.ef \\ de.af.bc \\ cf.bd.ae \end{matrix}\right\}, \quad \left\{\begin{matrix} ab.\overline{cd}.ef \\ de.af.bc \\ ac.be.df \end{matrix}\right\}, \quad \left\{\begin{matrix} ab.cd.\overline{ef} \\ de.af.bc \\ bf.ce.ad \end{matrix}\right\},$$

where the bar denotes the complete vertical column. We obtain then Mr. Kirkman's extension of Steiner's theorem:—*The Pascals intersect three by three, not only in Steiner's twenty points g, but also in sixty other points h.* The demonstration of Art. 267 applies alike to Mr. Kirkman's and to Steiner's theorem.

In like manner if we consider the triangles 1 and 5, the lines joining corresponding vertices are the same as for 1 and 4; therefore the corresponding sides intersect on a right line, as they manifestly do on a Pascal. In the same manner the corresponding sides of 4 and 5 must intersect on a right line, but these intersections are the three h points,

$$\left\{\begin{matrix} \overline{ab}.ce.df \\ de.bf.ac \\ cf.ae.bd \end{matrix}\right\}, \quad \left\{\begin{matrix} ae.\overline{cd}.bf \\ bd.af.ce \\ ac.be.df \end{matrix}\right\}, \quad \left\{\begin{matrix} ac.bd.\overline{ef} \\ df.ae.bc \\ ce.bf.ad \end{matrix}\right\}.$$

Moreover, the axis of 4 and 5 must pass through the intersection of the axes of 1, 4, and 1, 5, namely, through the g point, $\left.\begin{matrix} ab.cd.ef \\ de.af.bc \\ cf.be.ad \end{matrix}\right\}$.

In this notation the g point is found by combining the complete vertical columns of the three h points. Hence we have the theorem: "*There are twenty lines x, each of which passes through one g and three h points.*" The existence of these lines was observed independently by Mr. Cayley and myself. The proof here given is Mr. Cayley's.

Again, let us take three Pascals meeting in a point h. For instance,

$$\left.\begin{matrix} ab.ce.df \\ de.bf.ac \end{matrix}\right\}, \quad \left.\begin{matrix} de.bf.ac \\ cf.ae.bd \end{matrix}\right\}, \quad \left.\begin{matrix} cf.ae.bd \\ ab.df.ce \end{matrix}\right\}.$$

We may, by taking on each of these a point p, form a triangle whose vertices are (df, ac), (bf, ae), (bd, ce), and whose sides are, therefore,

$$\left.\begin{matrix} ac.bf.de \\ df.ae.cb \end{matrix}\right\}, \quad \left.\begin{matrix} bf.ce.ad \\ ae.bd.cf \end{matrix}\right\}, \quad \left.\begin{matrix} bd.ac.ef \\ ce.df.ab \end{matrix}\right\}.$$

Again, we may take on each a point h, by writing under each of the above Pascals $af.cd.be$, and so form a triangle whose sides are

$$\left.\begin{matrix} ac.bf.de \\ be.cd.af \end{matrix}\right\}, \quad \left.\begin{matrix} cf.ae.bd \\ be.cd.af \end{matrix}\right\}, \quad \left.\begin{matrix} df.ab.ce \\ be.cd.af \end{matrix}\right\}.$$

But the intersections of corresponding sides of these triangles, which must therefore be on a right line, are the three g points,

$$\left.\begin{matrix} be.cd.af \\ ac.bf.de \\ df.ae.bc \end{matrix}\right\}, \quad \left.\begin{matrix} be.cd.af \\ cf.ae.bd \\ ad.bf.ce \end{matrix}\right\}, \quad \left.\begin{matrix} be.cd.af \\ df.ab.ce \\ ac.ef.bd \end{matrix}\right\}, \quad \left.\begin{matrix} be.cd.af \\ cf.ab.de \\ ad.ef.bc \end{matrix}\right\}.$$

I have added a fourth g point, which the symmetry of the notation shows must lie on the same right line; these being all the g points into the notation of which $be.cd.af$ can enter. Now there can be formed, as may readily be seen, fifteen different products of the form $be.cd.af$; we have then Steiner's theorem, *The g points lie four by four on fifteen right lines*I.

My limits do not allow me to do more than add the enunciations of a few more theorems (principally Mr. Kirkman's), but the preceding examples are sufficient to show how they may be demonstrated, and how any reader who chooses to prosecute the study of the figure may find other theorems in great abundance: "*The twenty lines x pass four by four through fifteen points y.*" The four lines x whose g points in the preceding notation have a common vertical column will pass through the same point. "*There are sixty lines* J, *each of which passes through one*

point p and two points h." "*The lines* J *again pass three by three through sixty points j, three of which lie on each of the lines x.*" Mr. Kirkman calls points *m* the intersections of two Pascals, corresponding to hexagons which have four common sides, no opposite pairs being the same for both; for example, *abcdef*, *abcfed*; and points *r*, those corresponding to hexagons which have three common sides, two of which are contiguous; for example, *abcdef*, *abcefd*. "*The ninety points m lie three by three on sixty lines* M." "*There are sixty lines* R, *each containing six points r, and also one of the six points* P, *and which pass in threes through twenty points q.*"

ON THE PROBLEM TO DESCRIBE A CONIC UNDER CERTAIN CONDITIONS.

We saw (p. 120) that five conditions determine a conic; we can, therefore, in general describe a conic being given *m* points and *n* tangents where $m + n = 5$. We shall not think it worth while to treat separately the cases where any of these are at an infinite distance, for which the constructions for the general case only require to be suitably modified. Thus to be given *a parallel to an asymptote* is equivalent to one condition, for we are then given a point of the curve, namely, the point at infinity on the given parallel. If, for example, we were required to describe a conic given four points and a parallel to an asymptote, the only change to be made in the construction (p. 284) is to suppose the point E at infinity, and the lines DE, ME therefore drawn parallel to a given line.

To be given *an asymptote* is equivalent to two conditions, for we are then given a tangent and its point of contact, namely, the point at infinity on the given asymptote. To be given *that the curve is a parabola* is equivalent to one condition, for we are then given a tangent, namely, the line at infinity. To be given *that the curve is a circle* is equivalent to two conditions, for we are then given two points of the curve at infinity. To be given *a focus* is equivalent to two conditions, for we are then given two tangents to the curve (p. 237), or we may see otherwise that the focus and any three conditions will determine the curve; for by taking the focus as origin, and reciprocating, the problem becomes to describe a circle, three conditions being given; and the solution of this, obtained by elementary geometry, may be again reciprocated for the conic. Again, to be given *the pole, with regard to the conic, of any given right line*, is equivalent to two conditions; for three more will determine the curve. For (see figure, p. 134) if we know that P is the polar of $R'R''$, and that T is a point on the curve, T' the fourth har-

monic, must also be a point on the curve: or if OT be a tangent, OT′ must also be a tangent; if then, in addition to a line and its pole, we are given three points or tangents, we can find three more, and thus determine the curve. As a particular case of this, to be given *the centre* is equivalent to two conditions, for this is the pole of the line at infinity.

Given five points.—We have shown (Ex. 12, p. 284) how by the ruler alone we may determine as many other points of the curve as we please. We may also find the polar of any given point with regard to the curve; for we can perform the construction of p. 199 by the help of the same Example. Hence too we can find the pole of any line, and therefore also the centre.

Five tangents.—We may either reciprocate the constructions of Ex. 12, p. 284, or reduce this question to the last by p. 221.

Four points and a tangent.—We have already given one method of solving this question, p. 279. As the problem admits of two solutions, of course we cannot expect a construction by the ruler only. We may therefore apply Carnot's theorem (Art. 303),

$$Ac \cdot Ac' \cdot Ba \cdot Ba' \cdot Cb \cdot Cb' = Ab \cdot Ab' \cdot Bc \cdot Bc' \cdot Ca \cdot Ca'.$$

Let the four points a, a', b, b' be given, and let AB be a tangent, the points c, c' will coincide, and the equation just given determines the ratio $Ac^2 : Bc^2$, everything else in the equation being known. It may be remarked that this question may be reduced, if we please, to those which follow; for given four points, there are (Art. 307) three points whose polars are given; having also then a tangent, we can find three other tangents immediately, and thus have four points and four tangents.

Four tangents and a point.—This is either reduced to the last by reciprocation, or by the method given at the end of the last case; for given four tangents there are three points whose polars are given (Art. 235, Ex. 2).

Three points and two tangents.—This is solved by the help of the theorem (Art. 263), "Given two points and two tangents, the line joining the points of contact of the tangents passes through a fixed point;" or, more accurately, "through one or other of two fixed points." These fixed points are the foci of the system in involution, of which the two given points are one pair of conjugates, and the points where the line joining them is met by the given tangents are another; or we can easily find these points by Carnot's theorem, for let cc' be the given points, a and b the points of contact of the given tangents, and we have

$$Ac \cdot Ac' \cdot Ba^2 \cdot Cb^2 = Ab^2 \cdot Bc \cdot Bc' \cdot Ca^2.$$

But if the point where the chord of contact meets Ab be p we have, by the corresponding theorem, for a transversal meeting the sides of a triangle,

$$Ba^2 . Cb^2 . Ap^2 = Ab^2 . Ca^2 . Bp^2,$$

whence $Ap : Bp$ is determined.

If now, from considering the two given tangents and two of the given points, we learn that the chord of contact must pass through either of the points l, l', and from considering the first and third of the given points we find that the chord must pass through one of the points m, m', it is plain that the chord must be one of the four lines lm, lm', $l'm$, $l'm'$. The problem, therefore, has four solutions.

Two points and three tangents.—This may be solved as in the last case by the reciprocal of the theorem there employed, viz., "that the intersection of the tangents at the given points lies on one or other of two fixed lines;" or by the theorem itself, it is easy to see that the problem is reduced to describing a triangle whose sides pass through three given points, and whose vertices rest on three fixed lines.

To be given two points or two tangents to a conic is a particular case of being given that the conic has double contact with a given conic. For the problem to describe a conic having double contact with a given one, and touching three lines, or else passing through three points, see p. 283. Having double contact with two, and passing through a given point, or touching a given line, see p. 242. Having double contact with a given one, and touching three other such conics, see p. 256.

We have already alluded (p. 216) to the problem, "to describe a conic through four points to touch a given conic." Let the required conic be $S - kS'$, which is to touch S''. Then the polar of the point of contact, with regard to S'', is the tangent at the point, and is also its polar for $S - kS'$, and therefore passes through the intersection of the polars with regard to S and S'. Now let it be required to find the locus of a point such that its polars, with regard to S, S', S'', should meet in a point. If ξ, η, ζ be the current co-ordinates, we have to eliminate these between the equations of the three polars,

$$\xi \frac{dS}{dx} + \eta \frac{dS}{dy} + \zeta \frac{dS}{dz} = 0,$$

$$\xi \frac{dS'}{dx} + \eta \frac{dS'}{dy} + \zeta \frac{dS'}{dz} = 0,$$

$$\xi \frac{dS''}{dx} + \eta \frac{dS''}{dy} + \zeta \frac{dS''}{dz} = 0;$$

and the result is,

$$\frac{dS}{dx}\left(\frac{dS'}{dy}\cdot\frac{dS''}{dz}-\frac{dS'}{dz}\cdot\frac{dS''}{dy}\right)+\frac{dS}{dy}\left(\frac{dS'}{dz}\cdot\frac{dS''}{dx}-\frac{dS'}{dx}\cdot\frac{dS''}{dz}\right)$$
$$+\frac{dS}{dz}\left(\frac{dS'}{dx}\cdot\frac{dS''}{dy}-\frac{dS'}{dy}\cdot\frac{dS''}{dx}\right)=0,$$

a curve of the third degree, whose intersections with S″ give the six solutions sought. This solution can be performed by the help of a conic alone when S and S′ have double contact, and when the question is "to describe a conic having double contact with a given conic in two given points, and touching a given conic." Thus the point of contact of $S - L^2$ with S′ must lie on the locus of points whose polars with regard to S and S′ intersect on L. But it is easy to see that this locus is the same as the locus of the poles of L with regard to all conics of the form $S - kS'$, which we have already seen to be a conic; its intersection with S′ gives the four points required.

Art. 309, Page 267.

If we seek the condition that the right line $ax + by + c$ should touch a conic given by its general equation, the analytical problem is precisely the same as that solved in this article, and the condition is found by substituting in the equation of the reciprocal curve a, b, c, for $x, y, -k^2$.

Art. 315, Page 271.

Many of these examples may be more simply regarded as particular cases of the theorem that a transversal cutting the sides of an inscribed quadrilateral and the conic is cut in involution. Thus in Ex. 1, two pairs of vertices of the quadrilateral coincide; and we have "the points where a transversal meets two tangents and the conic belong to a system in involution, of which the point where it meets the chord is a focus." In Example 2, P is the other focus. In Example 3, T is the centre. In Example 7, C is the centre, and this example generalized gives us, "if an inscribed quadrilateral be cut by a parallel to an asymptote, $Ca \cdot Cc = Cb \cdot Cd$."

When the focus of a system in involution is at infinity, since the other focus must bisect the interval between any two conjugate points, the relation of the system in involution becomes simply $AB = A'B'$. Thus if the chord of contact be at infinity, we have the well-known theorem of a right line cutting the hyperbola and its asymptotes. Or again, if several conics pass through four given points, the asymptote to any of them is divided so that $AB = A'B'$.

On the Evolutes of Conics.

We had intended postponing all mention of the evolutes of conics until we come to treat of curves of higher dimensions. As their equations, however, can be easily formed, we add here the method of obtaining them. We define the evolute as the locus of the centre of curvature of the different points of a conic. The co-ordinates of the centre of curvature are found by subtracting from the co-ordinates of the point of the conic the projections of the radius of curvature upon each axis. Now evidently this radius is to its projection on y as the normal to the ordinate y. We find the projection, therefore, by multiplying the radius $\frac{b'^2}{p}$ by $\frac{y'}{N} = \frac{b'^2 y'}{b^2}$. The y of the centre of the curvature then is $\frac{b^2 - b'^2}{b^2} y'$. But $b'^2 = b^2 + \frac{c^2}{b^2} y'^2$, therefore the y of the centre of curvature $= \frac{b^2 - a^2}{b^4} y'^3$. In like manner its $x = \frac{a^2 - b^2}{a^4} x'^3$. Solving then for x' and y', and substituting these values in the equation of the conic, we get for the equation of the evolute $\left(\text{writing } \frac{c^2}{a} = A, \frac{c^2}{b} = B\right)$,

$$\frac{x^{\frac{2}{3}}}{A^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{B^{\frac{2}{3}}} = 1.$$

The y of the centre of curvature of a parabola is found in like manner by multiplying $\frac{N}{\sin^2\theta}$ (Art. 248) by $\frac{y'}{N} = \frac{y'}{\sin^2\theta}$, and subtracting this quantity from y', which gives $-\frac{y'}{\tan^2\theta} = -\frac{4y'^3}{p^2}$ (Art. 214).

In like manner its $x = x' + \frac{p}{2\sin^2\theta} = x' + \frac{p + 4x'}{2}$.

Solving then for x' and y', and substituting in the equation of the parabola, we have for the equation of the evolute,

$$27py^2 = 16(x - \tfrac{1}{2}p)^3;$$

or by transforming the origin to the point $(y = 0,\ x = \frac{1}{2}p)$,

$$27py^2 = 16x^3,$$

the equation of a curve called the *semi-cubical parabola.*

INDEX.

PAGE.

ANGLE.

Between two lines whose equations are given, ... 25
Between two lines given by a single equation, ... 66
Between pair of conjugate diameters, ... 155
Between focal radius vector and tangent, ... 165
Made with axes by conjugate diameters, how related, ... 152
Between asymptotes depends on eccentricity only, ... 148
Subtended at focus by tangent to a conic from any point, . . . 196
Subtended at limit points of system of circles, ... 265
Theorems respecting, how projected, ... 312, &c.

ANHARMONIC.

Pencils, fundamental theorem of, proved, ... 51
Ratio, what, when one point at infinity, ... 269
Ratio of four lines whose equations are given, ... 51, 52
Property of points on a conic, 218, 228, 262, 308
of tangents, ... 228, 262
These properties developed, . 271, &c.
Of four points on a conic when equal to that of four others on same conic, . . . 229, 231, 232
on a different conic, . . . 230, 282

PAGE.

ANHARMONIC—*continued.*

Of four points, equal that of their polars, ... 285
Of four diameters, equal that of their conjugates, ... 285
Of segments of tangent to one of three conics having double contact, by other two, . . . 308

ARC.

Line cutting off constant, how cut, 293
Theorems as to length of arcs of conics, ... 296

AREA.

Of a polygon, in terms of co-ordinates of its angles, ... 23
Constant, of triangle formed by joining ends of conj. diams., . 155
Constant between any tangent and asymptotes, ... 174
Constant, between parallels to asymptotes, ... 173
Of triangles equal, formed by drawing from end of each of two diams. a parallel to the other, 193
or by tangents at end of each, . 193
Found by infinitesimals, . . 290, &c.
Constant, cut from a conic by tangent to similar, ... 293
Line cutting off constant, bisected, 293

ASYMPTOTE.

A tangent through centre whose point of contact at infinity, . 140
Its own conjugate, ... 153

PAGE.

ASYMPTOTES—*continued.*
Diagonals of a parallelogram whose sides are conjugate diameters, 171
And pair of conjugate diameters form harmonic pencil, . . . 271
Portion of tangent between, bisected by curve, ... 172, 271
Equal intercepts on any chord between curve and, . . . 172, 328
Chords joining two fixed points to variable, intercept constant length on, ... 196, 268, 274
cut parallel to, in constant ratio, 272
Parallel to, bisected between any point and its polar, ... 270
how cut by two tangents and their chords, ... 273
Parallels to, through point on curve, include constant area, . 173, 219, 268, 274
how divide any semidiameter, . 273

AXES.
Of central conic found, ... 139
Bisect angle between asymptotes, . 141
Of a parabola, ... 177
Of reciprocal curve found, ... 265
Of similitude, ... 114, 203, 255

BRIANCHON'S THEOREM, 221, 253, 300

CARNOT'S THEOREMS OF TRANSVERSALS, 33, 263, 308

CENTRE.
Its co-ordinates, ... 129
Pole of the line at infinity, 141, 270, 310
Of similitude, ... 99, 203
Chords joining ends of radii through c. s. meet where, . . . 203, 221
Of system in involution, ... 277

CHASLES, M., ... 268, 269, 297

CIRCLE. (*See also* CONIC.)
Length and equation of tangent to, 83
Passes through two fixed points at infinity, ... 213

PAGE.

CIRCLE—*continued.*
Distance of any point of it from a chord, mean between its distances from tangents at its ends, ... 85
Product of perpendiculars on opposite sides of inscribed quadrilateral, equal, ... 86
Vectors from one end of diameter to ends of chord through fixed point on it, how cut tangent at other end, ... 93
Distances of each of two points from polar of other, are as their distances from centre, ... 95
Relation between four points on circle and an arbitrary point, . 102
Three circles meeting in a point whose other intersections lie in a right line, ... 100
Radical axis, centre, ... 107, &c.
Properties of circles having common radical axis, . . . 109, &c.
Circle touching three others, . 116, &c.
Tangents, area, and arc found by infinitesimals, ... 289

CONDITION THAT
Three points should be in a right line, ... 22
Three lines should meet in a point, 50
Four convergent lines should form a harmonic pencil, ... 51
A right line should pass through a fixed point, ... 60
Equation of second degree should represent a circle, ... 72
an ellipse, hyperbola, or parabola, 124
two right lines, 69, 126, 139, 216, 247
Equation of any degree should represent right lines, ... 70
through a given point, ... 71
should represent a curve through the origin, ... 73
Two circles should be concentric, . 72
Either axis touch a curve of second degree, ... 79, 125

PAGE.

CONDITION THAT—*continued.*

A given line touch a given conic, 150, 328
Two conics should touch, ... 216
Two conics should osculate, . . . 206
Conic inscribed in triangle touch another circumscribed, . . . 250
Two conics be similar, and similarly placed, ... 201
similar, and not similarly placed, 204

CONFOCAL CONICS.

Cut at right angles, ... 165
May be considered as inscribed in same quadrilateral, ... 237
Tangents from point on (1) to (2) equally inclined to tangent of (1), ... 166
Pole with regard to (2) of tangent to (1) lies on normal of (1), . 201
Used in finding axes of reciprocal curve, ... 265
Length of arc intercepted between tangents from, ... 296

CONIC, PROPERTIES OF. (*See also* ASYMPTOTE, FOCUS, &c.)

Rectangle under segments of parallel chords in constant ratio, 136
Equal diameters equally inclined to axis, ... 144
Product of perpendiculars on opposite sides of inscribed quadrilateral in constant ratio, . . 218
Reciprocal of this theorem, . . . 261
Anharmonic properties of, . . 218, 271
Parallel to either tangent, how cut by two tangents and their chord, ... 272
Reciprocals of squares of diameters at right angles, constant, . . 170
Rectangle under segments of two fixed parallel tangents by a variable, . . 191, 262, 268, 274
under segments of a variable between two parallels, ... 191

PAGE.

CONIC, PROPERTIES OF—*continued.*

Rectangle under segments of a variable between two conj. diams., 192
Chords of intersection of conic and circle equally inclined to axis, 208, 245
Tangent to circle having double contact with conic, varies as perp. on chord of contact, . . 217
Generalization of this theorem, . . 218
Chords of contact of two conics with common tangent meet on common chord, ... 219
Tangent through intersection of common chords cut harmonically, ... 279
Properties of chords through contact of two touching conics, . 255
Tangent to one cut harmonically where it meets other and chord of contact, ... 308
Properties of two conics having double contact with a third, . 219
Common tangent to two cut harmonically by third, ... 279
Properties of three conics having double contact with a fourth, 220, 255, &c.
of three conics having two points or tangents common, . 221, 255
If two conics have parallel axes, so have all through their intersection, ... 245
Three points which have same polars with regard to two conics, 266

CONJUGATE DIAMETERS, 131, 152
Expressions for their length, . 153, 159
Sum or difference of squares constant, ... 154
Angle between them, when least, 155
Parallel to supplemental chords, . 157
Construction for, ... 159, 172
Triangle formed by, constant, . . 155

CONJUGATE HYPERBOLÆ, . . 147

PAGE.

CONTINUITY, LAW OF, ... 204

CO-ORDINATES.
Of point cutting a given line in given ratio, ... 5, 57
Of intersection of two given right lines, ... 24
Of intersection of right line and circle or conic, . . . 77, 82, 135
Of points of contact of tangents to circle from given point, . . . 80
Of centre of conic, ... 129
Of centre of similitude of two circles, 112
Of centre of curvature of a conic, . 329
Transformation of (*see* TRANSFORMATION), ... 6
Trilinear, ... 224, &c.
Cartesian, a case of trilinear, . . . 226

CURVATURE.
Radius of, expressions for its length, 207, 294, &c.
Centre of, constructions for, . . . 208
Co-ordinates of, ... 329
Chord of, focal and central, . 209, 295

DEGREE.
Of an equation unaltered by transformation, ... 48
Of a curve determined by its intersections with a right line, . . 209

DIAMETER.
Pole of point at infinity on its ordinates, ... 270

DIRECTRIX (*see also* FOCUS), . . 167

DISTANCE.
Between two points in terms of their co-ordinates, ... 4, 9
Between variable point and fixed, when a rational function of the co-ordinates, ... 190

ECCENTRIC ANGLE, . . . 157, &c.
In terms of corresponding focal angle, ... 194

PAGE.

ECCENTRICITY, . . . 144, 146, 179
The same for similar conics, or whose asymptotes are parallel, 202
Of reciprocal hyperbola, depends on what, ... 264

ELLIPSE (*see also* CONIC), . . . 145
Origin of name, ... 171
Described by continuous motion, . 158, 175
Its area, ... 291

ENVELOPE, ... 228, 238
Of base of triangle whose sides pass through two fixed points, and vertices move on fixed lines, 254
and inscribed in a given conic, . 233 254, 309
given vertical angle and sum of sides, ... 240
subtending a constant angle at given point, two sides being given in position, ... 260
Of polar of fixed point with regard to a conic,
given four tangents, ... 255
given two tangents and two points, ... 246
having double contact with two given conics, ... 246
Of polar of centre of circle touching two given, ... 265
Of chord of a circle whose length is constant, ... 90
of conic subtending constant angle at focus, ... 237, 259
ditto, at any point of curve, . . 312
ditto, right angle at any point, . 260
joining $\mu \tan\phi$, $\mu \cot\phi$, . . . 231
making $\{ABCD\} = \{A'B'C'D'\}$, 232
such that $\{O.ATBT'\}$ is constant where TT' are points of contact of tangents from O, ... 315
where two fixed lines through one of four given points on conic meet curve again, ... 284

PAGE.

ENVELOPE—*continued.*
Of line such that $\Sigma\ m$ times perpendicular on it from several fixed points is constant, . . . 91
product of perpendiculars on it from two fixed points is constant, ... 239
its equation involves indeterminate in second degree, . . . 228
making a fixed angle with radius vector to a right line, through point where rad. vector meets line, ... 185, 239
ditto, for circle, ... 167, 260
Of tangents whose points of contact are in a right line to a series of conics having same focus and directrix (or touching two fixed lines in two given points), 259
Of asymptotes to a series of hyperbolæ having same focus and directrix, ... 259
Of one leg of given anharmonic pencil, while other three pass through fixed points, and vertex moves on a fixed line, . . 316
ditto, and vertex moves on conic through two of the points, . 316
Of side of polygon inscribed in conic, all the rest passing through fixed points, ... 282
or touching conics having double contact with given, ... 283
Of ellipse, given two conjugate diameters and sum of their squares, ... 240
Of dD where $\{abcd\} = \{ABCD\}$, and *abcd*, ABCD, on right lines, ... 287
Of CD where AC, BD are given in length and direction, and AB through fixed point, ... 286
$\frac{A}{\mu} + \frac{B}{\mu'} = 1$; $\mu + \mu' = C$, . . . 240
$(\mu A)^m + (\mu' B)^m + (\mu'' C)^m = 0$,
$(\mu a)^n + (\mu' b)^n + (\mu'' c)^n = 0$, 241

PAGE.

EQUATION,
Its meaning when co-ordinates of a point are substituted in it,
for a right line, ... 20
for a circle, ... 83
for a conic, ... 217
Of a right line,
through a given point, ... 20
through two given points, . . 21, 59
parallel to a given line; Cartesian and trilinear, ... 26, 58
through intersection of two given lines; Cartesian and trilinear, 48, 60
cutting angle between two lines in parts whose sines in given ratio, ... 51
bisecting angle between two given lines, ... 52
making a given angle with given line, ... 26
at infinity, ... 58
Polar equation of a right line, . . 27
Of a curve through *all* the intersections of two given curves, . . 48
Of axes of co-ordinates, ... 64
Of a pair of right lines, ... 65
Of bisectors of angles of two lines given by one equation, . . . 67
Of system of lines through origin, . 68
through any fixed point, . . . 68
Of a circle, ... 71
through three points, ... 101
circumscribing a triangle, . . . 102
inscribed in a triangle, ... 105
Polar equation of circle, ... 84
Chord of intersection of two circles, ... 106
Chord of contact of common tangents of two circles, ... 111
General, of second degree; Cartesian and trilinear, . . . 120, 224
referred to its axes, ... 143
Tangent to a circle or conic, . 76, 132, 149, 173, 180, 228
Polar of circle or conic, . 81, 127, 132, 150, 180, 230, 243

PAGE.

EQUATION—*continued.*
Pair of tangents from any point to conic, ... 83, 135
Diameter bisecting chords parallel to given line, ... 92, 129
Chord and tangent in terms of eccentric angle, ... 90, 160
Normal, ... 161
Polar equation of conic, centre pole, ... 144, 146
ditto, focus pole, . . 169, 187, 188
Hyperbola referred to asymptotes, . 148
Conic circumscribing a triangle, . 247
and having given centre, . . . 248
Conic inscribed in a triangle, . . 249
and having given centre, . . . 250
through five given points, . . . 222
touching five given lines, . . . 250
through intersection of two given conics, ... 209, 213
having double contact with a given conic, ... 210
having double contact with two given conics or circles, . 241, 242
having contact of third order with given conic, ... 210
touching two given lines at two given points, ... 214
such that each of three given lines is polar of intersection of two others, ... 215
inscribed in a quadrilateral, . . 242
Reciprocal of a conic, ... 266, 267

EVOLUTES OF CONICS, . . . 329

FAGNANI'S THEOREM, ... 297

FIXED POINT.
The following lines pass through a,
Base of a triangle, given vertical angle and sum of reciprocals of sides, ... 60
ditto, whose two sides pass through fixed points and vertices move on three convergent lines, ... 61, 254
ditto, if lines on which base

PAGE.

FIXED POINT—*continued.*
angles move, intersect on line joining the fixed points, . 62, 254
Polar of fixed point with regard to a circle, two points given, . . 109
ditto, with regard to a conic, four points given, . . . 200, 245, 255
Chord of intersection with fixed circle of circle through two given points, ... 108
ditto, with fixed conic, of conic through four points, two being on the fixed conic, ... 222
ditto, of two fixed lines with conic through four points, one lying on each of the fixed lines, 284
Chord of conic, given by trilinear equation, when? 231
bd, if $\{abcd\}$ be harmonic, . . . 314
subtending a right angle at fixed point on curve, ... 197
given bisector of angle it subtends at fixed point on curve, . . . 314
when product of tangents is constant of parts into which normal divides subtended angle, . 198
Line joining points of contact of circle touching two others, . 114
ditto, of two fixed tangents with conic passing also through two fixed points, . . . 220, 254, 326
ditto, of tangents to two conics which intersect on their common chord, ... 255
ditto, of a conic through two points having double contact with a given conic, ... 220
Line such that Σm times perpendicular on it from a number of fixed points $= 0$, ... 62
whose equation involves in first degree co-ordinates of point moving on a fixed line, . . . 63
co-efficients in whose equation are connected by relation of first degree, ... 64

PAGE.

FIXED POINT—*continued.*
Perpendicular on its polar, from point on fixed perpendicular to axis of conic, ... 168

FOCUS, ... 163
Infinitely small circle having double contact with conic, . . . 217
Intersection of tangents from two fixed imaginary points at infinity, ... 237
Focal properties, equations which give, ... 236
Distance of point on central conic from, ... 163
Sum or difference of focal distances constant, ... 163
This property generalized, . . . 242
Rectangle under focal radii, . . . 164
under focal perpendiculars on tangent, ... 165
This property reciprocated, . 261, 262
Focal radii equally inclined to tangent, ... 165, 290
Length of central parallel to focal radius vector, ... 166
Foot of perpendicular on tangent, its locus, ... 167
Focal distance varies as distance from directrix, ... 168
equal parallel to asymptote to meet directrix, ... 175
Focal chord, perpendicular to radius vector to its pole, . 168, 313
its segments have constant harmonic mean, ... 169, 261
varies as square of parallel diameter, ... 170
Sum of reciprocals of two rectangular focal chords constant, . . 170
Angle subtended by a chord, bisected by radius vector to its pole, 186, 237, 258
Constant angle subtended by intercept of variable tangent between two fixed, ... 187
Product of tangents constant of

PAGE.

FOCUS—*continued.*
half angles subtended by segments of chord through a fixed point, ... 317
or through point on conic having same focus and directrix, . . 317
Line joining intersection of focal normals and tangents goes through other focus, ... 197
ditto, and middle point of focal chord, parallel to axis, . . . 197
Length of intercept on any ordinate produced to meet focal tangent, 197
Point of contact with plane section of sphere inscribed in a right cone, ... 304

HARMONIC PROPERTY.
What, when one point at infinity, . 269
Of a quadrilateral, ... 54, 301
Of poles and polars, . 82, 134, 269, 272, 307
Pencil formed by two tangents and two co-polar lines, . . . 135, 270
by asymptotes and two conjugate diameters, ... 271
by diagonals of inscribed and circumscribed quadrilateral, . . 220
By chords of contact and common chords of two conics touching a third, ... 256

HEXAGON. (*See* BRIANCHON and PASCAL.)
Property of circumscribing, . . . 244

HYPERBOLA (*see* ASYMPTOTE and CONIC), ... 147
Origin of name, ... 171
Tangent from foot of ordinate to circle on transverse axis, in constant ratio to ordinate, . . 160
Equilateral; asymptotes cut at right angles, ... 154
Its area, ... 292

INFINITESIMALS, METHOD OF, ... 288, &c.

PAGE.

INFINITY.
Equation of line at, ... 58, 299
Its pole, ... 141, 270, 310
Polar of point at, ... 270
Direction of points at, on curve, how found, ... 123, 227
Line at, touches a parabola, . 211, 304
Two imaginary fixed points at, on every circle, ... 213

INVOLUTION, ... 275, &c.
Relation, what, when focus at infinity, ... 328

KIRKMAN'S THEOREMS, . 321, &c.

LIMIT POINTS.
Of a system of circles having common radical axis, ... 109
Circles through l. p. cutting the system orthogonally, ... 110
Reciprocals of system with regard to, confocal, ... 265
Theorems of angles subtended at, . 265

LOCUS.
Of vertex of triangle, given base and difference of squares of sides, ... 35
and vertical angle, ... 87
and ratio of sides, ... 87
and m times square of one side $+ n$ times square of other, . . 88
and sum or difference of sides, . 164
and difference of base angles, . . 189
and product of tangents of base angles, ... 190
ditto, of half base angles, . . . 190
and intercept by sides on fixed line, ... 281
Of vertex of triangle whose angles are given, one vertex fixed, and one moves along a right line, ... 45
of which one base angle has given magnitude and vertex, the other describes right line or circle, and rectangle under sides is given, ... 99

PAGE.

LOCUS—*continued.*
Of vertex of triangle whose sides pass through three fixed points, and base angles move on fixed lines, ... 235, 254, 280, 284
ditto, when the three fixed points are on a right line, . . . 42, 254
ditto, when two of them are on a line with intersection of the two fixed lines, ... 44, 254
ditto, when two of the sides are parallel to fixed lines, . . . 189
ditto, when base touches a conic touched also by fixed lines on which base angles move, 235, 280
ditto, or if base angles move on any conic having double contact with this conic, . . 235, 282
ditto, when base angles move on a conic passing through the two fixed points on the sides, 281
Vertex of a triangle whose sides touch a given conic, and base angles move on fixed lines, 232, 254
ditto, when base angles move one on each of two conics, base through cent. sim., and sides through fixed points, one on each of the conics, ... 282
Common vertex of several triangles, given bases and sum of areas, 44
vertex of right cone out of which given ellipse can be cut, . . 317
Of centre of (or of pole of a fixed line with regard to) a conic, given four points, 199, 245, 249, 255, 285, 309
given four tangents, . 200, 250, 255, 311
given the two foci, ... 200
given three points and a tangent, 249
given two points and two tangents, ... 245
given three tangents and a point, 250
given three points and parallel to asymptote, ... 286

PAGE.

LOCUS—*continued.*

Of centre of (or of pole of a fixed line with regard to) a conic having double contact with two given conics, ... 245
inscribed in a triangle, and touching another circumscribed to same, ... 250
Of centre of circle touching two given circles, ... 265
Of centre of inscribed circle, given base and sum of sides, . . . 190
Of pole with regard to one conic of tangent to another, ... 198
Of pole of line with regard to conic meeting it in two fixed points, and touching two conics which pass through these points, ... 310
Of points whose polars with regard to three given conics meet in a point, ... 328
Of intersection of tangents to two circles, in given ratio, . . . 107
whose sum or difference is constant, ... 242
to a conic, which cut at right angles, ... 185, 195, 259
to a parabola, which cut at a given angle, ... 238, 259
to a circle whose chord is of constant length, ... 91
to a conic whose chord passes through fixed point, . . . 94, 135
ditto, or subtends a constant angle at focus, ... 238, 259
from two points which cut a given line harmonically, . . . 313
ditto, when the line touches the conic, ... 314
ditto, and is cut in given anharmonic ratio, ... 315
from two points, each on one of four given tangents, ... 311
ditto, both on some one of four given tangents, ... 311

PAGE.

LOCUS—*continued.*

Of intersection of tangents at two fixed points on a conic also touching two fixed lines, 254, 310
ditto, also passing through another fixed point, and touching another fixed line, ... 311
to each of two conics, when line joining contacts passes through cent. sim. (through point of contact if they touch, through focus if common to both), . . 203, 220, 255

Intersection of lines joining transversely ends of base of a triangle and any parallel, . . . 37
transversely ends of parallels to sides of a parallelogram, . . 39
ends of parallel to base, to two fixed points on base, ... 40
ends of base, to ends of perpendiculars at proportional distances, ... 41

Intersection of perpendiculars to sides of triangle at ends of parallel to base, ... 39
perpendiculars at ends of line of constant length between two rectangular lines, ... 88
perpendiculars of a triangle given base and vertical angle, . . . 89
diagonals of a quadrilateral, two opposite sides fixed, and two others through fixed point, . 41
external bisector of vertical of a triangle, whose base and sum of sides are given, with perpendicular to either side at extremity of base, ... 46
ditto, for internal bisector and difference of sides, ... 46
corresponding sides of two given angles with fixed vertices, when intersection of other pair of corresponding sides moves on a right line, ... 280

PAGE.

LOCUS—*continued.*

Intersection of corresponding sides of two given angles with fixed vertices, when intersection of other pair of corresponding sides moves on a conic through the vertices of the angles, ... 281

ditto, if angles cut off constant intercepts on fixed lines, . . 281

focal radius vector, with corresponding eccentric vector, . . 193

eccentric vector, with corresponding normal, ... 193

tangent, with central rad. vec. parallel to vertical rad. vec. to contact, ... 194

projection of this theorem, . . . 310

PD, pd, where $\{abcd\} = \{ABCD\}$; P p fixed points, ... 287

Mean (arithmetic, harmonic, or geometric) of radii vectores to a circle, ... 97

Harmonic mean, radii vectores to a system of right lines, ... 47

Arithmetic and geometric mean, to two right lines, ... 189, 273

Extremity of reciprocal of perpendicular from fixed point on line through fixed point, . . 45

perpendicular from focus on tangent, ... 167, 184

ditto, or making constant angle with tangent, ... 197

polar subtangent (focus being pole), ... 170

radius vector proportional to radius vector of circle, ... 98

inversely proportional to radius vector of circle or line, . . . 99

Point cutting in given ratio, parallel chords of a circle, ... 145

line of constant length between two others, ... 158, 189

line cutting off constant area from a given angle, ... 190

PAGE.

LOCUS—*continued.*

Point cutting in given ratio chord of a circle of given length, . 91

intercept between two fixed lines of parallel to a given, . . . 36

intercept between two tangents of variable tangent to conic, . 315

Of point cutting in given anharmonic ratio chords of conic through fixed point, ... 310

on perpendicular, at height from base equal a side, given base and sum of sides, ... 35

such that sum of squares of perpendiculars on sides of an equilateral triangle be constant, . 86

whose distance from fixed line varies as distance from fixed point, ... 188

ditto, or as square of distance, . 88

such that triangle formed by joining feet of perpendiculars on sides of triangle has given area, ... 103

such that $\Sigma mr^2 =$ constant, . . 88

such that square of tangent to circle is as product distances from two lines, ... 217, 262

or tangent as distance from line, 217

Middle points of rectangles inscribed in triangle, ... 36

of parallel chords of a conic, 92, 129

of convergent chords of a circle, 97

Point of contact of tangents from given point to concentric circles, ... 259

Point on parallel meeting sides of triangle so that $oc^2 = oa.ob$, . 273

at which two given portions of same line subtend equal angles, 275

Of O, if line joining O to pole of AO pass through given point, 314

Of one point of a line cut in given anharmonic ratio, while the other three describe right lines, and the line itself touches a

PAGE.

LOCUS—*continued.*
conic touched by two of the right lines, ... 315
Focus of parabola, given three tangents, ... 187, 260

MAC LAURIN'S MODE OF GENERATING CONICS, 235, 254, 280, 284

NEWTON'S MODE OF GENERATING CONICS, ... 280

NORMAL.
Its equation and length, ... 161
Rectangle under segments of normal equal square semi-conjugate diameter, ... 162
under normal and central perpendicular = square of semiaxis, . 162

NUMBER.
Of terms in general equation of n^{th} degree, ... 70
Of conditions that it should represent right lines, ... 70
Of conditions which determine a conic, ... 120
Of points in which a line meets a conic, ... 121
Of tangents which can be drawn to a conic, ... 125
Of intersection of two given curves, 10 205, 209

OSCULATION OF CONICS, . . 206

PARABOLA.
Its figure, ... 178
described by continuous motion, 183
Origin of name, ... 171
Its area, ... 290
An ellipse, whose focus given, axis major infinite, eccentricity = 1, 178
Its diameters parallel, ... 130
meet curve at infinity, ... 131
Touched by line at infinity, . 211, 214, 264, 304

PAGE.

PARABOLA—*continued.*
Rectangle under segments of a chord varies as intercept on diameter, ... 138
Projection of two points on axis = intercept made by their polars, ... 180
Point where tangent meets axis, and point of contact, equidistant from focus, ... 183
Tangent makes equal angles with axis and focal radius vector, . 183
Angle between two tangents, half angle between focal radii to contacts, ... 186, 238
Tangents from point of directrix at right angles, ... 237
Diameter bisected between any point and its polar, ... 270
Diameter how divided by line joining two variable points to a fixed, ... 272
by a chord and its two tangents, 273
Difference of ordinates constant where two variable tangents meet a fixed, ... 268
A variable tangent, how divided by three fixed, ... 274
Equal and similarly placed have contact of third order, . . . 213

PARAMETER OF A CONIC, . . 169
Constant for reciprocals of equal circles, ... 261
Constant for sections of right cone at same distance from vertex, 304
Of diameter of parabola, four times focal distance, ... 181, 182

PASCAL'S THEOREM.
Proved, ... 223, 283, 309
Reciprocated, ... 253
Theorems connected with the hexagon, ... 224, 321, &c.

PERPENDICULAR.
From point on line, its length, equation, and sign, ... 19, 20

PAGE.

PERPENDICULAR—*continued.*
From centre on tangent of conic, 155, 156
From focus on tangent, . . . 164, 184
From focus on asymptote, ... 174
From centre and foci on polar of any point, ... 195, 196

POLAR. (*See* EQUATIONS.)
Of point found by linear construction, ... 94, 199
Polar subtangent, ... 170
Polar of a given line with regard to a circle, ... 81
Polar parallel to tangent and to conjugate diameter, ... 133
Chord parallel to polar, bisected, 128, 270
If A be on polar of B, B on polar of A, ... 133
Harmonic property of pole and polar, ... 134, 269
Lines joining vertices of triangle and polar triangle meet in a point, ... 96, 243
Intersections of corresponding sides on right line, ... 255
Given four points or four tangents; three points, and their polars are given, ... 199
Three points whose polars the same with regard to two conics, . 266
A point and its polar equivalent to two conditions, ... 325

PONCELET, M.,. 251, 297

PROBLEMS.
Find a point such that product of perpendiculars may be constant, on a given line from ends of chord of circle through it, ... 92
Find centre of a given conic, . . 130
polar of a given point (linear construction), ... 94, 199
axes (ditto, given two conjugate diameters), ... 144, 192
asymptotes (ditto, given two conjugate diameters), . . . 148, 172

PAGE.

PROBLEMS—*continued.*
Draw normal from any point, 161, 166, 195
Describe a circle touching three others, ... 118, 265
a conic through four points to touch a line, ... 279
ditto, a conic,. 216, 327
ditto, having double contact with given conic, and single with three others also having double contact, ... 256
ditto, and touching three given lines, ... 283
To describe a conic, given five conditions, ... 325, &c.
touching one conic, and having double contact with another at two given points, ... 328
Given five points on conic, find tangent at any, ... 223
where line through any, meets curve again, ... 284
find centre, ... 284
Given five tangents, find point of contact of any, ... 221
Inscribe in a conic a triangle, sides through three fixed points, . 234, 255, 284
Circumscribe a triangle whose vertices on three fixed lines, . . 255
Inscribe polygon, sides through fixed points, ... 283
Given two pairs of conjugates of system in involution, find conjugate of any other, ... 276
the centre, the foci, ... 277

PROJECTIONS, METHOD OF, 297, &c.
Projective theorems, ... 300
Orthogonal projection, ... 317

QUADRILATERAL.
Middle points of its diagonals in right line, ... 32
Harmonic properties of, . 54, 301, 321
Inscribed in a circle, ... 86

PAGE.

QUADRILATERAL—*continued.*
Inscribed in a conic, . . 199, 218, 309
Its sides and diagonals cut transversal in involution, ... 278
Properties of circumscribed, . 199, 261
Diagonals of inscribed and circumscribed form harmonic pencil, 220

RADICAL AXIS, CENTRE, 107, 222, 255

RATIO.
Co-ordinates of point where line is cut in a given, ... 5, 57
In which the line joining two given points is cut by a given line, . 33
ditto, by the line joining two given points, ... 34
ditto, by a given circle or conic, 82, 135

RECIPROCAL POLARS, . . 251, &c.
Degree in general of recipr. polar, 253
Species, axes, eccentricity, how determined, ... 264
Equation of reciprocal of a conic, . 267
ditto, transformed to new origin, 267
Reciprocals with regard to a parabola, ... 268

RECTIFICATION OF CONICS, . 296

SIMILAR FIGURES, ... 201, &c.
Conics have common chord at infinity, ... 211
Parabolæ touch at infinity, . . . 212
Tangent to one cuts constant area from other, ... 293
Similar anharmonic systems, . . . 278

STEINER'S THEOREM, . . 224, 329

TRANSVERSAL (*see* CARNOT).
How cuts side of triangle, ... 33

TRANSFORMATION OF CO-ORDINATES.
Formulæ for, ... 6
Cannot alter degree of equation, . 48
Of general equation of second degree, ... 122

PAGE.

TRANSFORMATION OF CO-ORDINATES—*continued.*
Highest powers unaltered by, . . 122
Absolute term how formed, . . . 122
Alters not sign of $B^2 - 4AC$, . . 143
Of equation of reciprocal curve to new origin, ... 267

TRIANGLE.
Perpendiculars meet in a point, 29, 50, 59
Bisectors of sides meet in a point, . 30, 31, 58
Bisectors of angles meet in a point, 53
Perpendiculars at middle points of sides meet in a point, . . . 32
Lines joining vertices to contacts of inscribed circle or conic, meet in a point, 96, 104, 105, 249
Relation between segments, in which sides are cut by a transversal, 33
ditto, by lines joining assumed point to vertices, ... 34, 301
These six points in involution, . . 278
If through vertices be drawn three lines meeting in a point, so will three equally inclined to sides, 53
Feet of perpendiculars on sides from point of circumscribing circle in one right line, ... 103
Inscribed in conic, radius of circumscribing circle, ... 318
If lines joining vertices of two triangles meet in a point, intersections of opposite sides in one right line, ... 55, 301
Three points in a right line where each side is met by line joining intersections of two non-corresponding sides, ... 56
If perpendiculars from vertices of one on sides of second meet in a point, so will perpendiculars from vertices of second on sides of first, ... 59
Axes of three co-polar triangles meet in a point, ... 321

THE END.

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